Statistical Field Theory
Dr M B Wingate

Mathematical Tripos, Part III
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Examples Sheet 1

1. Consider a modification of the Ising model where the spin degrees-of-freedom take the values $\sigma_n = 1, 0, -1$. One might refer to this as the spin-1 Ising model, and to the $\sigma_n = \pm 1$ case as the spin-$\frac{1}{2}$ Ising model (with a factor $\frac{1}{2}$ absorbed somewhere). Here we solve the $D = 1$ spin-1 Ising model. Start by showing that the partition function is
$$Z = \mathrm{Tr}\, W^N$$
where W is the 3×3 matrix
$$W = \begin{pmatrix} z\mu^2 & \mu & z^{-1} \\ \mu & 1 & \mu^{-1} \\ z^{-1} & \mu^{-1} & z\mu^{-2} \end{pmatrix}$$
with the shorthand $z = e^{\beta J}$ and $\mu = e^{\beta h/2}$. For the case $h = 0$ show that this matrix can be expressed in the form $W = P\Lambda P^{-1}$ where
$$\Lambda = \begin{pmatrix} 2\cosh\beta J & \sqrt{2} & 0 \\ \sqrt{2} & 1 & 0 \\ 0 & 0 & 2\sinh\beta J \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}.$$
Hence find the eigenvalues of W and show that in the thermodynamic limit the free energy of the system is
$$F = -NT \log\left\{\frac{1}{2}\left[1 + 2\cosh\beta J + \sqrt{(2\cosh\beta J - 1)^2 + 8}\right]\right\}.$$

2. By considering $\langle\sigma_0\rangle$ in equilibrium show that in the mean field approximation to the Ising model in D-dimensions the equilibrium magnetization is given by the solution to
$$m = \tanh\beta(qJm + h)$$
where $q = 2D$. The free energy in this approach is
$$\mathcal{A} = -T\log[2\cosh\beta(qJ\overline{m} + h)] + \frac{1}{2}qJ\overline{m}^2\,.$$
Show that the expression for the equilibrium magnetization above can also be obtained by minimizing $\mathcal{A}$ with respect to $\overline{m}$. Below we use $\mathcal{F}$ to denote the equilibrium free energy.

   The critical exponent α governs the divergence in the specific heat as $T \to T_c$
$$C \sim |T - T_c|^{-\alpha} \quad \text{where} \quad C = T\left.\frac{\partial^2\mathcal{F}}{\partial T^2}\right|_{h=0}.$$

Using the expression for m above, show that the $h = 0$ free energy in equilibrium is

$$\mathcal{F} = -T \log 2 + \frac{1}{2} T \log(1 - m^2) + \frac{1}{2} qJm^2 \,.$$

By assuming the expansion $m^2 = c_1 t + c_2 t^2 + \ldots$ where $t \equiv (T - T_c)/T_c$, derive that

$$\mathcal{F} = -T \log 2 + \frac{1}{2} T \log(1 - c_1 t - c_2 t^2) + \frac{1}{2} qJ(c_1 t + c_2 t^2) \,.$$

By expanding $\mathcal{F}$ in t show that

$$\mathcal{F} = a_0 + a_1 t + \frac{1}{2} a_2 t^2$$

for some constants a_i, and consequently that

$$C = \frac{a_2}{T_c} + O(t)$$

and hence that the exponent $\alpha = 0$.

3. In the Blume-Capel model in D-dimensions the spins σ_n take values $\sigma_n = 1, 0, -1$. The Hamiltonian is an extension of the Ising-like one discussed on the previous sheet

$$H = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j + g \sum_i \sigma_i^2 - h \sum_i \sigma_i$$

where the first sum is over all nearest-neighbour pairs and the other sums are over all sites. Say each site has q nearest neighbours. Use the mean field approach to show that the free energy density of this system is approximated by

$$\mathcal{A} = \frac{1}{2} Jq\overline{m}^2 - T \log[1 + 2\kappa \cosh \beta(Jq\overline{m} + h)]$$

where $\kappa = e^{-\beta g}$ and $\overline{m}$ is the magnetization. *[Hint: do not approximate the* $g \sum_i \sigma_i^2$ *term.]*

For $h = 0$ expand $\mathcal{A}$ as a power series in $\overline{m}$. For what values of (T, κ) does mean field theory predict (i) ordinary critical behaviour, (ii) tricritical behaviour, (iii) a first order transition? In each case find the value of the critical temperature $T_c(\kappa)$.

Calculate the critical exponent α for both critical and tricritical behaviours.

4. The q-state Potts model is a generalisation of the Ising model. At each lattice site lives a variable $\sigma_i \in \{1, 2, \ldots, q\}$. The Hamiltonian is given by the sum over nearest neighbours

$$H_{\text{Potts}} = -\frac{3J}{2} \sum_{\langle ij \rangle} \delta_{\sigma_i \, \sigma_j}$$

How many ground states does the system at at $T = 0$?

Show that the 3-state Potts model is equivalent to the Hamiltonian

$$H = -J \sum_{\langle ij \rangle} \vec{s}_i \cdot \vec{s}_j$$

where $\vec{s}_i$ take values in the set

$$\vec{s}_i \in \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix} , \begin{pmatrix} -1/2 \\ -\sqrt{3}/2 \end{pmatrix} \right\}$$

By developing a mean field theory for H determine the self-consistency requirement for the magnetisation $\vec{m} = \langle \vec{s}_i \rangle$. Compute the mean field free energy and show that the system undergoes a first order phase transition even in the absence of an external field.

[Hint: This calculation will be simpler if you argue that you can focus on magnetisation vectors of the form $\vec{m} = (m, 0)$.]

5. The free energy $\mathcal{A}$ of an Ising system with variable order parameter $\overline{m}$ is given by

$$\mathcal{A} = -h\overline{m} + \mathcal{A}_2\overline{m}^2 + \mathcal{A}_4\overline{m}^4 + \mathcal{A}_6\overline{m}^6 \, ,$$

where it is assumed that $\mathcal{A}_6 > 0$ and that $\mathcal{A}_2$ and $\mathcal{A}_4$ are functions of external fields T and g, with $\mathcal{A}_2 \sim T - T_c(g)$ and where h is the applied magnetic field.

On dimensional grounds argue that at equilibrium $\mathcal{F}$ may be expressed as

$$\mathcal{F} = \frac{|\mathcal{A}_2|^{3/2}}{\mathcal{A}_6^{1/2}} \, \Phi \left(\frac{\mathcal{A}_4}{2\sqrt{|\mathcal{A}_2|\mathcal{A}_6}}, \frac{h\mathcal{A}_6^{1/4}}{|\mathcal{A}_2|^{5/4}} \right) ,$$

where $\Phi(0, 0)$ is finite and nonzero.

Compare this expression with the generic form for the free energy near the tricritical point, namely

$$\mathcal{A} = |T - T_c(\tilde{g})|^{2-\alpha} \, \Phi \left(\frac{\tilde{g}}{|T - T_c(\tilde{g})|^{\phi}}, \frac{h}{|T - T_c(\tilde{g})|^{\Delta}} \right)$$

where $\tilde{g} \propto \mathcal{A}_4$ and $\tilde{g}$ has been substituted for g as one of the independent external fields. Deduce that

$$\alpha = \frac{1}{2}, \quad \phi = \frac{1}{2}, \quad \Delta = \frac{5}{4} .$$

Define the critical temperature at the tricritical point to be $T_{TCP} \equiv T_c(\tilde{g} = 0)$.

(a) For $h = 0$ consider the trajectory in $(T, \tilde{g})$ space defined by the limit $\tilde{g} \to 0$ and $T \to T_{TCP}$ keeping the following ratio fixed

$$x \equiv \frac{\tilde{g}}{|T - T_c(\tilde{g})|^{\phi}} .$$

Observe that $\mathcal{A} \sim |T - T_{TCP}|^{3/2}$, and therefore $\alpha = \frac{1}{2}$. The trajectory lies in the tricritical region, i.e., we see tricritical exponents as we approach the transition. ϕ is known as the crossover exponent since it controls the shape of the trajectory and hence defines the boundary of the tricritical region.

(b) For $h = 0$, $\tilde{g}$ fixed, and $T \to T_c$ show that $\alpha = 0$, (i.e. normal critical behaviour) as long as it can be assumed that the function G defined by

$$y\, G(y, 0) = \Phi\left(\frac{1}{y}, 0\right)$$

is finite and nonzero at $y = 0$.

The crucial point is that to use dimensional analysis the existence of scaling functions such as Φ and G must be assumed and that these functions are finite and nonzero when their arguments are set to zero.