Statistical Field Theory | Mathematical Tripos, Part III
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Examples Sheet 2

1. If you have not already done so, fill in the steps of the lecture notes to show that the RG equations corresponding to our decimation transformation for the $D = 1$ Ising model are

$$x' = \frac{x(1+y)^2}{(x+y)(1+xy)}$$
$$y' = \frac{y(x+y)}{1+xy}$$
$$w' = \frac{w^2xy^2}{(1+y)^2(x+y)(1+xy)}$$

where $x = e^{-4\beta J}$, $y = e^{-2\beta h}$, and $w = e^{4\beta K}$. From linearizing near the fixed point at $(x, y) = (0, 1)$, we found eigenvalues of the linearized RG transformation and the associated critical exponents. Use these to deduce that the singular part of the free energy per spin satisfies

$$f(x, \rho) = b^{-1} f(b^2 x, b\rho)\,,$$

for a scale change $b = 2$ and where $y = 1 - \rho$. Use this result to show that

$$f(x, \rho) = \sqrt{x}\,\tilde{f}(\rho/\sqrt{x})\,,$$

where $\tilde{f}(z) = f(1, z)$. Verify that this is consistent with the exact result for the free energy if we choose

$$\tilde{f}(z) = -T\sqrt{1 + \frac{z^2}{4}}$$

for the singular part.

2. Continuing with the $D = 1$ Ising model, compare the complete expression for the free energy $\mathcal{F}$, derived in the notes, with its scaling form. What plays the role of the inhomogeneous part?

   For fixed J and $h = 0$ find an expression as $T \to 0$ for the leading singularity in the analogue of the specific heat $C = \partial^2\mathcal{F}/\partial t^2$, where $t = x$. Comment on what this implies for the value of α and the validity of the scaling relation $\alpha + 2\beta + \gamma = 2$.

   Suppose now $T > 0$ and held constant with $h = 0$. Let $J \to \infty$. What is the value of α in this case?

3. Consider a one-dimensional lattice of N sites. On each site is a degree-of-freedom σ_i ($i = 1, 2, \ldots, N$) which can take any of q integer values, say $\sigma_i = 0, 1, \ldots, q-1$. Impose periodic boundary conditions so that $\sigma_{i+N} = \sigma_i$. Let the Hamiltonian be

$$H(J) = -J\sum_i \delta_{\sigma_i\sigma_{i+1}}$$

with $J > 0$ and $\delta_{\alpha\beta}$ the Kronecker δ-function. This is the q-state Potts model.

Carry out a decimation type of renormalization group transformation. That is, consider

$$\exp\left[-H(J) - NK\right]$$

where K is independent of the degrees-of-freedom $\{\sigma_i\}$. Sum over the σ_i on every other site to obtain exact recursion relations for J and K. You may find it convenient to define $x = e^J$ and $w = e^K$ and use these to write the recursion relations.

Show that $J = 0$ and $J = \infty$ are fixed points of the renormalization group transformation and find whether each is stable or unstable.

4. (Binney *et al.,* 5.2) Consider the $D = 2$ Ising model on a square lattice with lattice spacing a, and with nearest-neighbour (i.e. distance a apart) and next-to-nearest-neighbour (i.e. distance $a\sqrt{2}$ apart) interactions, respectively corresponding to the first and second terms in the Hamiltonian

$$H = -K \sum_{\langle ij \rangle} \sigma_i \sigma_j - L \sum_{\langle\langle k\ell \rangle\rangle} \sigma_k \sigma_\ell \,.$$

Perform a thinning of degrees-of-freedom by summing over the spins, "decimating" the spins, on every second site, in a "checker-board" fashion – say, decimate spins with (x, y) coordinates $(2m, 2n{+}1)$ and $(2m{+}1, 2n)$, with m and n integers. Rescale the lattice by a factor $b = \sqrt{2}$ to recover the original lattice spacing. Calculate the interactions on the blocked lattice keeping only terms up to $O(K^2)$ and $O(L)$. Show that there are only 2 such interactions to this order, and that they are the same operators as the original ones, with new couplings

$$K' = 2K^2 + L$$
$$L' = K^2 \,.$$

Find the critical points for these RG equations and identify the nontrivial fixed point. Linearizing about this point, find a value for the critical exponent ν.

Sketch the RG flows in the K-L plane, assuming positive K and L.