Examples Sheet 3

1. Consider the Landau-Ginzburg Hamiltonian on a lattice with N sites.

$$H(\{\phi_i\}) = a^D \sum_{i=1}^{N} \left[\frac{1}{2} \sum_{\mu=1}^{D} (\Delta_\mu \phi_i)^2 + \frac{1}{2} r_0 \phi_i^2 + \frac{u_0}{4!} \phi_i^4 \right]$$

The forward difference operator Δ_μ is defined to be such that

$$\Delta_\mu \phi_i \;\equiv\; \frac{1}{a}[\phi(\mathbf{x}_i + a\hat{\mu}) - \phi(\mathbf{x}_i)]\,.$$

Make the Landau approximation that the thermodynamics can be described by the saddle-point approximation to the partition function.

Consider the broken phase, where $m = \langle \phi_0 \rangle \neq 0$. First introduce a external, constant magnetic field h and discuss the relative importance of the field configuration $-\phi_0$ as $h \to 0$ and $N \to \infty$. Does the order in which these limits are taken matter? Why, or why not?

2. (Adapted from Le Bellac 2.5)

(a) Consider real scalar field ϕ on a (hyper)cubic lattice in D dimensions with unit lattice spacing ($a = 1$) and Hamiltonian

$$H \;=\; -\,J \sum_{\langle ij \rangle} \phi_i \phi_j$$

where $\langle ij \rangle$ denotes nearest neighbours. Let the probability for a particular configuration of field variables $[\phi_i]$ be

$$P[\phi_i] \;=\; \frac{1}{Z} \exp\left[-\beta H \;-\; \frac{1}{2} r_0 \sum_i \phi_i^2 \right]\,.$$

By going to Fourier space and introducing an external magentic field as a source, show that the position-space, connected correlation function $\langle \phi_j \phi_k \rangle_c$ is given by

$$\langle \phi_j \phi_k \rangle_c \;=\; \int \frac{d^D q}{(2\pi)^D} \frac{e^{i\mathbf{q}\cdot(\mathbf{x}_j - \mathbf{x}_k)}}{r_0 - 2\beta J \sum_{\mu=1}^{D} \cos q_\mu}$$

where $q_\mu \in (-\pi, \pi]$ is the μ-component of $\mathbf{q}$.

(b) Now take as the Hamiltonian (restoring the explicit lattice spacing a)

$$H \;=\; a^D \sum_{i=1}^{N} \left[\frac{1}{2} \sum_{\mu=1}^{D} (\Delta_\mu \phi_i)^2 \;+\; \frac{1}{2} m^2 \phi_i^2 \right]$$

where $i \in [1, N]$ labels the sites. Show or argue that the correlation function in momentum space is given by

$$\tilde{G}(\mathbf{q}) = \left[\frac{2}{a^2} \sum_{\mu=1}^{D} (1 - \cos(aq_\mu)) + m^2 \right]^{-1} .$$

[You need not repeat all the steps that are in your solution to (a).] Show that if $aq_\mu \ll 1$ (for all μ) the correlation function is nearly rotationally invariant. For small aq_μ, what is the largest term which breaks rotational invariance?

3. In the Gaussian model, let us write the Hamiltonian as

$$H[\phi] = \int d^D x \left[\frac{1}{2} (\nabla\phi)^2 + \frac{1}{2} r_0 \phi^2 \right] .$$

The momentum-space 2-point correlation function is

$$\tilde{G}_0(p) = \frac{1}{p^2 + r_0} .$$

We wish to show that

$$G_0(x) = \int \frac{d^D p}{(2\pi)^D} e^{-ip\cdot x} \frac{1}{p^2 + r_0} \sim \begin{cases} \dfrac{1}{r^{D-2}} & r \ll \xi \\ \dfrac{\xi e^{-r/\xi}}{(r\xi)^{(D-1)/2}} & r \gg \xi \, . \end{cases}$$

where $r = |x|$. The $r \ll \xi$ limit may be obtained straightfowardly by making a sensible approximation for this limit. The $r \gg \xi$ limit requires more work. The most elegant solution makes use of the identity that, for $u > 0$,

$$u = \int_0^\infty dt \, e^{-t/u} .$$

Then another, more common trick can be employed to carry out the momentum integral. One last approximation yields the desired form.

Another approach is to pick out one of the D dimensions. Let $x = (s, \mathbf{x})$, where $\mathbf{x}$ is a $D - 1$ dimensional vector. Then show that

$$\int d^{D-1} x \, G_0(s, \mathbf{x}) \propto \xi e^{-|s|/\xi}$$

and proceed from there, making a suitable approximation for the $r \gg \xi$ limit.

4. (Le Bellac 3.3) Consider the Landau-Ginzburg Hamiltonian with additional interaction terms:

$$H = \int d^D x \left[\frac{1}{2\alpha} (\nabla\phi)^2 + \frac{1}{2} r_0 \phi^2 + \frac{u_0}{4!} \phi^4 + \frac{u_6}{6!} \phi^6 + \frac{u_8}{8!} \phi^8 + \frac{v_0}{4!} \phi^2 (\nabla\phi)^2 \right] .$$

For a renormalization group transformation where short wavelength modes are integrated out, derive renormalization group equations to first order in $V = H - H_0$, where H_0 is the Hamiltonian of the Gaussian model. For $D > 4$, verify that all the interaction terms in the Hamiltonian are irrelevant. Also calculate the modification to the term with $(\nabla\phi)^2$.

5. (Le Bellac 3.9) The generalization from a single scalar field to an n-component scalar field is straightforward; the interacting Hamiltonian can be written

$$H_1 = \int d^D x \left[\frac{1}{2} \sum_{j=1}^{n} (\nabla \phi_j)^2 + \frac{1}{2} r_0 \sum_{j=1}^{n} \phi_j^2 + \frac{u_0}{4!} \left(\sum_{j=1}^{n} \phi_j^2 \right)^2 \right] .$$

Let us add another term to H_1, arriving at a new Hamiltonian

$$H_2 = H_1 + \frac{v_0}{4!} \int d^D x \sum_{j=1}^{n} \phi_j^4 .$$

Show that H_2 is positive definite only if both the following conditions are satisfied

$$u_0 + v_0 > 0 \quad \text{and} \quad u_0 + \frac{v_0}{n} > 0 .$$

Show that the set of parameters (u_0, v_0) is equivalent to the set $(u_0 + \frac{3}{2} v_0, -v_0)$ for the case that $n = 2$.

For general n, and defining $u = u_0/8\pi^2$ and $v = v_0/8\pi^2$, one can derive the following differential renormalization equations (you can take them as given):

$$\begin{aligned} \frac{du}{d \log b} &= \epsilon u - \frac{n+8}{6} u^2 - uv \\ \frac{dv}{d \log b} &= \epsilon v - 2uv - \frac{3}{2} v^2 \end{aligned}$$

where $\epsilon = 4 - D$. Show that these equations imply the existence of four fixed points. [These are termed the Ising, Heisenberg, Gaussian, and cubic fixed points. See if you can decide which is which, giving reasons.] For $\epsilon > 0$ study the stability of these fixed points in the (u, v) plane and sketch the renormalization group flow in this plane. Show that one must distinguish between the cases $n < 4$ and $n > 4$.