Example Sheet 1

1. The four dimensional 4×4 Dirac matrices are defined uniquely up to an equivalence by $\{\gamma^\mu,\gamma^\nu\}=2g^{\mu\nu}1$, with 1 the unit matrix. We may also require that if $\gamma^\mu=(\gamma^0,\boldsymbol{\gamma})$ then $\gamma^{\mu\dagger}=(\gamma^0,-\boldsymbol{\gamma})$. If $[X,\gamma^\mu]=0$ for all μ then $X\propto 1$ and if γ^μ,γ'^μ both obey the Dirac algebra then $\gamma'^\mu=S\gamma^\mu S^{-1}$ for some S. Define the charge conjugation matrix C by $C\gamma^{\mu\,T}C^{-1}=-\gamma^\mu$, where T denotes the matrix transpose. Show that $[C^TC^{-1},\gamma^\mu]=0$ and hence that $C^T=cC,\ c=\pm 1$. Derive the results

$$\big(\gamma^\mu C\big)^T=-\,c\gamma^\mu C\,,\quad \big(\gamma^5 C\big)^T=c\gamma^5 C$$
$$\big(\gamma^\mu\gamma^5 C\big)^T=c\gamma^\mu\gamma^5 C\,,\quad \big([\gamma^\mu,\gamma^\nu]C\big)^T=-c[\gamma^\mu,\gamma^\nu]C\,.$$

Hence, since there are 6 independent antisymmetric and 10 symmetric 4×4 matrices, show that we must take $c=-1$.

Using the assumed hermeticity properties of the Dirac matrices, show $[\gamma^\mu,CC^\dagger]=0$ so that we may take $CC^\dagger=1$.

The matrix B is defined by $B\gamma^{\mu *}B^{-1}=(\gamma^0,-\boldsymbol{\gamma})$. Show that $B\gamma^{5*}B^{-1}=\gamma^5$. With the assumed form for $\gamma^{\mu\dagger}$ verify that we may take $B=\pm\gamma^5 C$.

*Generalise the above argument for finding c to $2n$ dimensions when the Dirac matrices are $2^n\times 2^n$ and we may take as a linearly independent basis 1 and $\gamma^{\mu_1\cdots\mu_r}=\gamma^{[\mu_1}\ldots\gamma^{\mu_r]}$, where [ ] denotes antisymmetrisation of indices, for $r=1,\ldots 2n$ ($\gamma^{\mu_1\cdots\mu_r}$ has $\binom{2n}{r}$ independent components). Show that $C(\gamma^{\mu_1\cdots\mu_r})^TC^{-1}=(-1)^{\frac12 r(r+1)}\gamma^{\mu_1\cdots\mu_r}$ and hence $c=(-1)^{\frac12 n(n+1)}$. Generalise γ^5 by taking $\hat\gamma=i^{n-1}\gamma^0\gamma^1\ldots\gamma^{2n-1}$ and show that $\hat\gamma$ is hermitian and $\hat\gamma^2=1$. Show that

$$\psi^c=C\bar\psi^T\,,\quad \psi'=\hat\gamma\psi\quad\Rightarrow\quad \psi=-c\,C\bar\psi^{c\,T}\,,\quad \psi'^c=-(-1)^n\hat\gamma\psi^c\,.$$

In what dimensions is possible to have Majorana-Weyl spinors, so that $\psi^c=\pm\psi'=\psi$?

2. A Dirac quantum field transforms under parity so that

$$\hat P\psi(x)\hat P^{-1}=\gamma^0\psi(x_P)\,,\quad x_P^{\,\mu}=(x^0,-\mathbf{x})\,,$$

and has an interaction with a scalar field $\phi(x)$

$$\mathcal{L}_I(x)=g\,\bar\psi(x)\psi(x)\phi(x)+g'\bar\psi(x)i\gamma^5\psi(x)\phi(x)\,.$$

Obtain the necessary form for $\hat P\phi(x)\hat P^{-1}$ to ensure that the theory is invariant under parity if $g'=0$. What are the transformation properties of $\phi(x)$ for parity invariance when $g=0$? Can parity be conserved in a theory if both g,g' are non zero?

How does the axial current $j_5^{\,\mu}(x)=\bar\psi(x)\gamma^\mu\gamma^5\psi(x)$ transform under parity?

3. For a free operator Dirac field $\hat{\psi}(x)$ assume $\hat{\psi}(x) = \sum_r a_r \, \psi_r(x)$ where $\{\psi_r(x)\}$ forms a basis for solutions of the Dirac equation and a_r are operators. Explain why a basis may be chosen so that $B\psi_r^*(x) = \psi_{r'}(x_T)$ where $x_T^\mu = (-x^0, \mathbf{x})$ and $B\gamma^{\mu *}B^{-1} = (\gamma^0, -\boldsymbol{\gamma})$. Assume the time reversal operator is defined so that $\hat{T}a_r\hat{T}^{-1} = a_{r'}$. What is $\hat{T}\hat{\psi}(x)\hat{T}^{-1}$?

4. Under charge conjugation and time reversal a Dirac field ψ transforms as

$$\hat{C}\psi(x)\hat{C}^{-1} = C\bar{\psi}^T(x) \,, \qquad \hat{T}\psi(x)\hat{T}^{-1} = B\psi(x_T) \,, \quad x_T^\mu = (-x^0, \mathbf{x}) \,.$$

with $\hat{C}$, $\hat{T}$ the unitary, anti-unitary operators implementing these operations (note, if $\hat{T}|\phi\rangle = |\phi_T\rangle$ then $\langle\phi'|\phi\rangle = \langle\phi_T|\phi'_T\rangle$). The matrices C, B are defined in question 1; also note $C^\dagger C = B^\dagger B = 1$. Show that, if X is matrix acting on Dirac spinors,

$$\hat{C}\,\bar{\psi}(x)X\psi(x)\hat{C}^{-1} = \bar{\psi}(x)X_C\psi(x) \,, \quad \hat{T}\,\bar{\psi}(x)X\psi(x)\hat{T}^{-1} = \bar{\psi}(x_T)X_T\psi(x_T) \,,$$

where $X_C = CX^TC^{-1}$ (take ψ and $\bar{\psi}$ to anti-commute) and $X_T = BX^*B^{-1}$. Hence determine the transformation properties under charge conjugation and time reversal of

$$\bar{\psi}(x)\psi(x) \,, \qquad \bar{\psi}(x)i\gamma^5\psi(x) \,, \qquad \bar{\psi}(x)\gamma^\mu\gamma^5\psi(x) \,.$$

If $|\pi\rangle$ is a boson with momentum p and $\langle 0|\bar{\psi}(0)i\gamma^5\psi(0)|\pi(p)\rangle \neq 0$ show that, in a theory in which P and C are conserved, then the boson must have negative intrinsic parity and also positive charge conjugation parity.

5. From Maxwell's equation $\partial_\nu F^{\mu\nu} = e\bar{\psi}\gamma^\mu\psi, F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ derive the required transformation properties of $A_\mu(x)$ to ensure invariance under parity, charge conjugation and time reversal. Show that $\int d^4x \, \epsilon^{\mu\nu\sigma\rho}F_{\mu\nu}F_{\sigma\rho}$ is odd under both P and T.

6. For a Dirac field ψ define $\psi_\pm = \frac{1}{2}(1 \pm \gamma^5)\psi$. Show that $\bar{\psi}_\pm\gamma^5 = \mp\bar{\psi}_\pm$. Let $\Psi_\pm = \begin{pmatrix}\psi_\pm \\ C\bar{\psi}_\mp^T\end{pmatrix}$ and show that then $\bar{\Psi}_\pm = (\bar{\psi}_\pm, -\psi_\mp^T C^{-1})$. [Hint: it is easier to keep the new 2-dimensional "super-spin" space separate from the 4-dimensional spinor space of ψ.] A generalized Lorentz-invariant mass term can be written as

$$\mathcal{L}_m = \frac{1}{2}\Psi_+^T C^{-1}\mathcal{M}\Psi_+ \; - \; \frac{1}{2}\bar{\Psi}_+\mathcal{M}^* C\bar{\Psi}_+^T$$

where $\mathcal{M}$ is a symmetric 2×2 matrix which communtes with C and γ^μ. [The notation can be confusing, but it is convensional. You can read the matrices more explicitly as $\mathbb{1}_4 \otimes \mathcal{M}$, $C \otimes \mathbb{1}_2$ and $\gamma^\mu \otimes \mathbb{1}_2$]. Verify that $\mathcal{L}_m^\dagger = \mathcal{L}_m$. If $\mathcal{M} = \begin{pmatrix}0 & m \\ m & 0\end{pmatrix}$ show that by absorbing any phase into $\psi_\pm$ we can take m real and positive, and that this reduces to the conventional Dirac mass term $\mathcal{L}_m = -m\bar{\psi}\psi$. Show that the kinitic term $\mathcal{L}_K = \bar{\psi}i\partial\!\!\!/\psi = \bar{\Psi}_+ i\partial\!\!\!/\Psi_+$. Regarding Ψ_+ and $\bar{\Psi}_+$ as independent and assuming $\mathcal{L} = \mathcal{L}_K + \mathcal{L}_m$ derive the equations

$$i\partial\!\!\!/\Psi_+ - \mathcal{M}^* C\bar{\Psi}_+^T = 0, \quad i\partial\!\!\!/ C\bar{\Psi}_+^T - \mathcal{M}\Psi_+ = 0$$

and hence that the mass-squared eigenvalues are found by solving $\det(p^2 1 - \mathcal{M}^* \mathcal{M}) = 0$.

Requiring $\hat{T}\psi(x)\hat{T}^{-1} = B\psi(x_T)$ and $\hat{T}\bar{\psi}(x)\hat{T}^{-1} = \bar{\psi}(x_T)B^{-1}$, with $B = \gamma^5 C$ as in question 1, show that $\hat{T}\Psi_+(x)\hat{T}^{-1} = B\Psi_+(x_T)$. Hence demonstrate that $\mathcal{M}$ should be real in order to have $\hat{T}\mathcal{L}_m(x)\hat{T}^{-1} = \mathcal{L}_m(x_T)$.

If $\mathcal{M} = \begin{pmatrix} 0 & m \\ m & M \end{pmatrix}$ with m real and positive and $|M| \gg m$, show that the masses are approximately $|M|$ and $m^2/|M|$.