Example Sheet 3

1. The Lagrangian density for the Weinberg-Salam theory with gauge fields $\mathbf{W}_\mu, B_\mu$, a complex scalar field ϕ and fermion fields ψ may be written as

$$\begin{aligned}\mathcal{L} \;&=\; -\tfrac{1}{4}\mathbf{F}^{\mu\nu}\cdot\mathbf{F}_{\mu\nu} - \tfrac{1}{4}B^{\mu\nu}B_{\mu\nu} + (D^\mu\phi)^\dagger D_\mu\phi - \tfrac{1}{4}\lambda(\phi^\dagger\phi - \tfrac{1}{2}v^2)^2 + \bar{\psi}i\gamma^\mu D_\mu\psi\,,\\ &\quad -\big(\bar{\psi}\Gamma_2\phi\,\tfrac{1}{2}(1+\gamma^5)\psi_2 + \bar{\psi}\Gamma_1\phi^c\,\tfrac{1}{2}(1+\gamma^5)\psi_1 + \text{hermitian conjugate}\big)\,,\end{aligned}$$

where $\boldsymbol{\sigma}$ are the usual Pauli matrices, Γ_1 and Γ_2 are Yukawa couplings, and

$$\mathbf{F}_{\mu\nu} = \partial_\mu\mathbf{W}_\nu - \partial_\nu\mathbf{W}_\mu - g\mathbf{W}_\mu\times\mathbf{W}_\nu\,,\quad B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu\,,$$

$$\psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}\,,\quad \phi = \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}\,,\quad \phi^c = i\sigma_2\phi^*\,,\quad D_\mu\phi = \partial_\mu\phi + i(g\mathbf{W}_\mu\cdot\tfrac{1}{2}\boldsymbol{\sigma} + g'B_\mu Y)\phi\,,$$

$$D_\mu\psi = \partial_\mu\psi + i(g\mathbf{W}_\mu\cdot\tfrac{1}{2}\boldsymbol{\sigma} + g'B_\mu y)\tfrac{1}{2}(1-\gamma^5)\psi + ig'B_\mu(y+Y\sigma_3)\tfrac{1}{2}(1+\gamma^5)\psi\,.$$

Identify the gauge group of $\mathcal{L}$ and show how each term in $\mathcal{L}$ is gauge invariant (note that Y only enters in the product $g'Y$ so that its value is essentially arbitrary). Show that a mass term for the gauge fields of the form

$$m_W^2 W^{\mu\dagger}W_\mu + \frac{1}{2}m_Z^2 Z^\mu Z_\mu$$

is produced where $W_\mu = \frac{1}{\sqrt{2}}(W_{1\mu} - iW_{2\mu})$ and $Z_\mu = \cos\theta_W W_{3\mu} - \sin\theta_W B_\mu$ for a suitable choice of the angle θ_W. How is m_Z/m_W related to θ_W? Explain why it is impossible to introduce mass terms for the fermion fields into $\mathcal{L}$ compatible with gauge invariance. What are the fermion charges which give the coupling to the photon?

2. Show that the decay rate of a Higgs boson to a $\ell\bar{\ell}$ pair is given by

$$\Gamma_{H\to\ell\bar{\ell}} = \frac{G_F}{\sqrt{2}}\frac{1}{4\pi}\frac{m_\ell^2}{m_H^2}\big(m_H^2 - 4m_\ell^2\big)^{\frac{3}{2}}\,,$$

assuming $m_H > 2m_\ell$.

3. For two generations of quarks with masses m_d, m_s, m_u, m_c, suppose the Lagrangian contains mass terms is given by $\mathcal{L}_m = -\big(\overline{q}_+ m_+\tfrac{1}{2}(1+\gamma_5)q_+ + \overline{q}_- m_-\tfrac{1}{2}(1+\gamma_5)q_- + \text{h.c.}\big)$ with

$$q_+ = \begin{pmatrix}u'\\ c'\end{pmatrix}\,,\quad q_- = \begin{pmatrix}d'\\ s'\end{pmatrix}\,,\quad m_+ = \begin{pmatrix}0 & a\\ a^* & b\end{pmatrix}\,,\quad m_- = \begin{pmatrix}0 & c\\ c^* & d\end{pmatrix}\,,\ (b, d\,\text{real})\,.$$

where u', d', c', s' are quark fields (the prime indicates they diagonalize the weak charged current interaction). Given $R(\theta) = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$, define θ_+ by

$$R(\theta_+)\begin{pmatrix}0 & |a|\\ |a| & b\end{pmatrix}R(\theta_+)^{-1} = \begin{pmatrix}m_u & 0\\ 0 & -m_c\end{pmatrix}$$

and define θ_- similarly for m_s and m_d. Show that after suitable rephasing of quark fields, one can use R to diagonalize $m_\pm$. Hence, from the mixing matrix so-generated in the quarks' weak charged-current interactions, show that the Cabbibo angle in this case is given by

$$\theta_C = \theta_- - \theta_+ = \tan^{-1}\sqrt{\frac{m_d}{m_s}} - \tan^{-1}\sqrt{\frac{m_u}{m_c}} \ .$$

4. In the μ decay process,

$$\mu^-(p) \to e^-(k) + \bar{\nu}_e(q) + \nu_\mu(q') \ ,$$

suppose the initial μ is polarised. The spin polarisation may be represented by a 4-vector s^μ, with $s \cdot p = 0$, where in the muon rest frame $s^\mu = (0, \mathbf{s})$, and the corresponding Dirac spinor then satisfies $u_\mu(p)\overline{u}_\mu(p) = (\gamma \cdot p + m_\mu)\frac{1}{2}(1 + \gamma^5\gamma \cdot s)$. Show that if $\mathcal{M}$ is the matrix element of $\mathcal{L}_W(0)$ for this process then

$$\sum_{\text{spins } e,\bar{\nu}_e,\nu_\mu} |\mathcal{M}|^2 = 64 G_F^{\,2}\, q' \cdot k \ q \cdot (p - m_\mu s) \ .$$

Hence obtain, neglecting the electron mass, for the differential decay rate in the μ rest frame for the final electron of energy E_e emitted into a solid angle $\mathrm{d}\Omega(\hat{\mathbf{k}})$ about the direction $\hat{\mathbf{k}}$

$$\mathrm{d}\Gamma = \frac{G_F^{\,2} m_\mu^5}{24(2\pi)^4}\, x^2\big(3 - 2x + (2x-1)\,\hat{\mathbf{k}} \cdot \mathbf{s}\big)\, \mathrm{d}x\, \mathrm{d}\Omega(\hat{\mathbf{k}}) \, .$$

where $x = 2E_e/m_\mu$, $0 \le x \le 1$. Explain why this decay distribution is a direct indication of the breakdown of parity invariance.

5. If $A_\alpha(x)$ is $\Delta S = \Delta Q = 1$ weak hadronic current then the kaon decay constant F_K is defined by

$$\langle 0 | A_\alpha(0) | K^-(p) \rangle = i\sqrt{2} F_K p_\alpha \ .$$

Use the theory of weak decays to show the decay rate for a kaon decaying to a muon and antineutrino is

$$\Gamma_{K^- \to \mu^- + \bar{\nu}_\mu} = \frac{G_F^{\,2} F_K^2 \sin^2\theta_C}{4\pi} m_\mu^2 m_K \left(1 - \frac{m_\mu^2}{m_K^2}\right)^2 \ .$$

6. Show that the interaction of the Z boson with a lepton field ℓ can be written as

$$\mathcal{L}_I = \frac{g}{2\cos\theta_W}\, \bar{\ell}\gamma^\mu(v - a\gamma^5)\ell\, Z_\mu \ ,$$

where for the electron $v = 2\sin^2\theta_W - \frac{1}{2}, a = -\frac{1}{2}$ while for ν_e then $v = a = \frac{1}{2}$. Calculate the decay rate $\Gamma_{Z \to \ell\bar{\ell}}$ neglecting the lepton mass.

7. Show that including the Z as well as the photon the differential cross section for $e^-e^+ \to \ell\bar{\ell}$ where $\ell = e, \mu, \tau$ has the form, neglecting lepton masses,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{4q^2}\Big\{(1+\cos^2\theta)\big(1+2v^2D+(v^2+a^2)^2D^2\big)+4\cos\theta\big(2a^2D+v^2a^2D^2\big)\Big\}\ ,$$

where $v = 2\sin^2\theta_W - \frac{1}{2}, a = -\frac{1}{2}$ and $D = \frac{G_F}{\sqrt{2}}\frac{m_Z^2}{q^2-m_Z^2}\Big/\frac{2\pi\alpha}{q^2}$. [The differential cross section here is the cross section per unit solid angle, $\mathrm{d}\Omega = \sin\theta\mathrm{d}\theta\mathrm{d}\phi$.] For $q^2 \approx m_Z^2$ the total cross section behaves like

$$\sigma_{e^-e^+\to\ell\bar{\ell}} \sim 12\pi\,\frac{\Gamma_{Z\to e^-e^+}\Gamma_{Z\to\ell\bar{\ell}}}{(q^2-m_Z^2)^2+m_Z^2\Gamma^2}\ ,$$

where Γ is the total decay width. Show that the formula for the differential cross section is compatible with the result for $\Gamma_{Z\to\ell\bar{\ell}}$.