Standard Model — Mathematical Tripos, Part III
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Example Sheet 4

1. In QCD let $a = g^2/(4\pi)^2$. The running coupling $a(\mu^2)$ is defined by

$$\mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} a = \beta(a)\,, \qquad \beta(a) = -\beta_0 a^2 - \beta_1 a^3 - \beta_2 a^4 + O(a^5)\,.$$

Show that for a suitable choice of Λ

$$\frac{1}{a(\mu^2)} = \beta_0 \log \frac{\mu^2}{\Lambda^2} + \frac{\beta_1}{\beta_0} \log\log \frac{\mu^2}{\Lambda^2} + O\left(1 \Big/ \log \frac{\mu^2}{\Lambda^2}\right).$$

If the coupling is redefined so that

$$\bar{a} = f(a) = a + v_1 a^2 + v_2 a^3 + O(a^4)\,,$$

show that $\bar{\beta}(\bar{a}) = \beta(a) f'(a)$ and hence $\bar{\beta}_0 = \beta_0$, $\bar{\beta}_1 = \beta_1$, $\bar{\beta}_2 = \beta_2 + \beta_0(v_2 - v_1^2) - \beta_1 v_1$. If $\bar{a}(\mu^2) = f(a(\mu^2))$ is written in terms of $\bar{\Lambda}$ in the same form as a in terms of Λ above show that

$$\log \frac{\bar{\Lambda}^2}{\Lambda^2} = \frac{v_1}{\beta_0}\,.$$

Suppose a physical observable R has a perturbative expansion in terms of the coupling $a(\mu^2)$ of the form $R = a^N\big[r_0 + r_1 a + r_2 a^2 + \dots\big]$. Under the above redefinition $r_1 \to \bar{r}_1 = r_1 + N v_1 r_0$, $r_2 \to \bar{r}_2 = r_2 + N v_2 r_0 + \frac{1}{2}N(N-1)v_1^2 r_0 + (N+1)v_1 r_1$. Show that

$$\hat{r}_1 = r_1 - \beta_0 N r_0 \log \frac{\mu^2}{\Lambda^2}\,, \qquad \hat{r}_2 = r_2 + N\frac{\beta_2}{\beta_0}\, r_0 - \frac{N+1}{2N}\,\frac{r_1^2}{r_0} - \frac{\beta_1}{\beta_0}\, r_1\,,$$

are invariant under redefinitions of the couplings.

2. Use the interaction $\mathcal{L}_W = -G_F/\sqrt{2}\, J_\alpha^{\mathrm{had}\dagger}\, \bar{\nu}_\tau \gamma^\alpha (1-\gamma_5)\tau$ to show that the total decay rate for $\tau^- \to \nu_\tau + \mathrm{hadrons}$ is

$$\Gamma_{\tau^- \to \nu_\tau + \mathrm{hadrons}} = \frac{G_F^2 m_\tau^3}{32\pi^2} \int_0^{m_\tau^2} \mathrm{d}\sigma \left(1 - \frac{\sigma}{m_\tau^2}\right)^2 \left(\rho_0(\sigma) + \left(1 + \frac{2\sigma}{m_\tau^2}\right)\rho_1(\sigma)\right),$$

where

$$\sum_X (2\pi)^4 \delta^4(P_X - k) \langle 0 | J_\alpha^{\mathrm{hadrons}} | X \rangle \langle X | J_\beta^{\mathrm{hadrons}\dagger} | 0 \rangle = k_\alpha k_\beta \rho_0(k^2) + (-g_{\alpha\beta}k^2 + k_\alpha k_\beta)\rho_1(k^2)\,,$$

and X covers all possible hadronic final states. If X is restricted to the π^- show that $\rho_0(\sigma) = 4\pi F_\pi^2 \cos^2\theta_C \delta(\sigma - m_\pi^2)$ and $\rho_1(\sigma) = 0$. Hence find $\Gamma_{\tau^- \to \nu_\tau + \pi^-}$.

3. (From Donoghue, Golowich, Holstein, Chapter IV) In describing the decay $\mu \to e\gamma$, one may try to use an effective Lagrangian $\mathcal{L}_{3,4}$ which contains terms of dimension 3 and 4

$$\mathcal{L}_{3,4} \;=\; a_3(\bar{e}\mu + \bar{\mu}e) \;+\; ia_4(\bar{e}\not{D}\mu \;+\; \bar{\mu}\not{D}e)$$

where $D_\mu = \partial_\mu + ieQA_\mu$ and a_3, a_4 are constants. Show by direct calculation that $\mathcal{L}_{3,4}$ does not lead to $\mu \to e\gamma$. If $\mathcal{L}_{3,4}$ is added to the QED Lagrangian for muons and electrons, show that one can define new fields μ' and e' to yield a Lagrangian which is diagonal in flavour. Thus, even in the presence of $\mathcal{L}_{3,4}$ there are two conserved fermion numbers. Finally, at dimension 5, show that $\mu \to e\gamma$ can be described by including in the Lagrangian

$$\mathcal{L}_5 \;=\; \bar{e}\sigma^{\mu\nu}(c + d\gamma^5)\mu F_{\mu\nu} \;+\; \text{h.c.}$$

where c and d are constants and $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$.

4. Construct the effective Lagrangian of the Goldstone bosons associated with the spontaneous symmetry breaking $SU(2) \times U(1) \to U(1)$.

5. Work through the notes to compute the hadronic contribution to $\pi^- \to e^-\bar{\nu}_e$ using the leading order chiral Lagrangian. What does this imply about the dimensionful parameter F?