A BRIEF INTRODUCTION TO VORTEX DYNAMICS AND TURBULENCE

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The emphasis in this short introductory chapter is on those fluid dynamical phenomena that are best understood in terms of convection and diffusion of vorticity, the curl of the velocity field. Vorticity is generated at fluid boundaries, and diffuses into the fluid where it is subject to convection, stretching and associated intensification. Far from boundaries, viscous effects may be negligible, and then vortex lines are transported with the fluid. Vortex rings, which propagate under their own self-induced velocity, are a widely observed phenomenon, and a fundamental ingredient of fluid flow. Stretching and intensification is best illustrated by the 'Burgers vortex' (the simplest model for a hurricane) in which these process are in equilibrium with viscous diffusion. Instabilities of Kelvin-Helmholtz type are all-pervasive in highly sheared flow, and inexorably lead to transition to turbulence. In turbulent flow, the vorticity is random, but these fundamental processes still dictate many features of the flow. Fully three-dimensional turbulence is characterised by a cascade of energy through a broad spectrum from large scales to very small scales at which kinetic energy is dissipated by viscosity, a scenario that leads to the famous (-5/3) Kolmogorov spectrum. These topics are reviewed and discussed with a view to geophysical applications. The phenomena of intermittency and concentrated vortices as revealed by direct numerical simulation are also briefly discussed.

1. Introduction

Vortex (or vorticity) dynamics is concerned with the manner in which swirling flows evolve in fluids when viscous (i.e. internal friction) effects are relatively weak, and can be neglected in a first approximation. Such flows are controlled largely by inertial effects. An understanding of vortex dynamics is an essential preliminary to a consideration of turbulent flows in which the vorticity distribution is a highly complex function of position. Its time evolution is most easily understood through the statement that "vortex lines are frozen in the fluid", i.e. they are transported with the flow like material curves of fluid particles. This is not quite the whole story however, because, insofar as the flow may be treated as incompressible, the vorticity is intensified as the vortex lines are transported, in proportion to the stretching of vortex line elements. This stretching is very persistent in a turbulent flow, leading to very strong intensification of vorticity coupled with progressive decrease of the scale of variation of the flow, an effect usually described in terms of an 'energy cascade'. This cascade to small scales is ultimately controlled by viscosity, no matter how weak this physical property of the fluid may be; and one of the remarkable properties of turbulent flow is that the rate of dissipation of energy by viscosity is independent of the value of viscosity even in the limit as this tends to zero, and this because the smallest scales of the flow adjust in just such a way as to dissipate the kinetic energy at the very rate at which it cascades down from larger scales.

The central role of vorticity in describing fluid motion was recognised by Hermann von Helmholtz (1858), who first recognised the above crucial 'frozen-in' property. The 150th anniversary of the publication of this seminal paper was marked by the IUTAM Symposium *150 years of Vortex Dynamics*, recently held at the Technical University of Denmark (Aref 2010; the 50 papers contained in this volume provide an indication of the huge current scope and applications of the subject). The theory of vorticity was taken up and enthusiastically developed by William Thomson (later Lord Kelvin) (1867; 1869 and many subsequent papers), who proposed that the atomic structure of the various elements might be explained in terms of knotted vortex tubes, whose 'knottedness' would be conserved under frozen field evolution. Such structures turn out to be dynamically unstable, and Kelvin was ultimately obliged to abandon his theory of 'vortex-atoms'; nevertheless, his pioneering investigations opened up the new field of hydrodynamic instability, providing important clues concerning the ubiquity of turbulent, as opposed to laminar, flows in all large-scale natural systems. Figure 1 shows Helmholtz and Kelvin around 1870, when both were at the height of their powers and creativity.

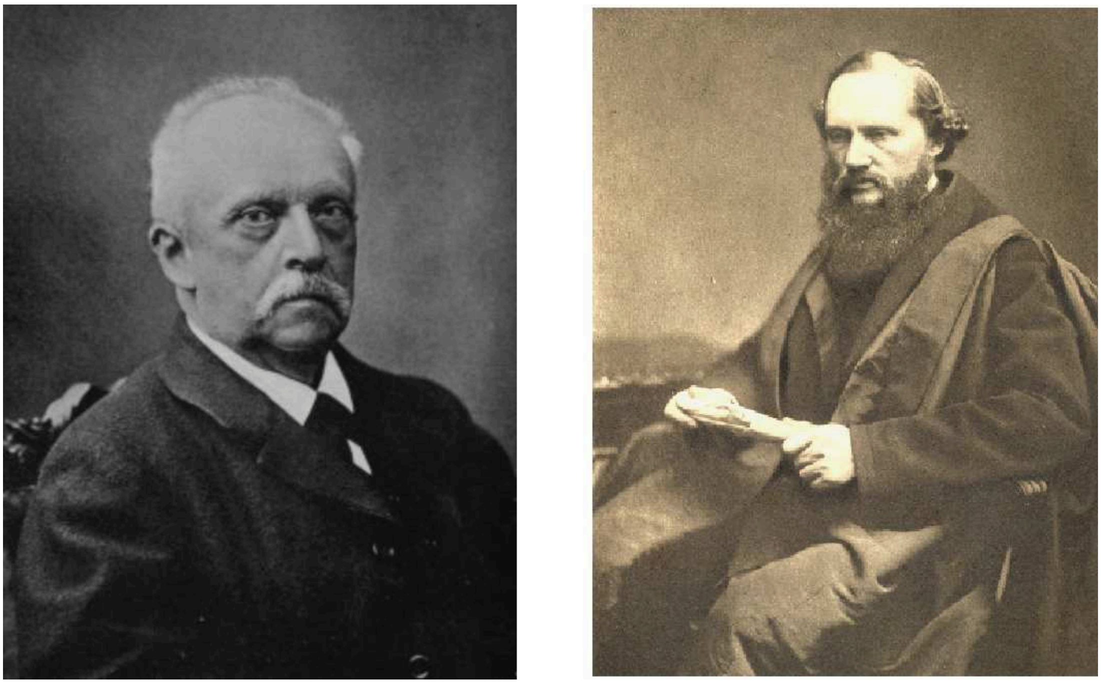

Fig. 1. Hermann von Helmholtz (left) and William Thomson (Lord Kelvin): the early pioneers of vortex dynamics.

2. Vorticity and the Biot-Savart law

Let $\mathbf{u}(\mathbf{x}, t)$ be the velocity field in a fluid which fills all space. This is of course an idealisation, relevant when we consider fluid behaviour that is uninfluenced by remote fluid boundaries. We shall suppose further, for simplicity, that the fluid has uniform density ρ, and that it (or rather the flow) is incompressible, i.e. $\nabla \cdot \mathbf{u} = 0$. Under this approximation, sound waves are filtered out of the governing Navier-Stokes equations. The vorticity field $\boldsymbol{\omega}(\mathbf{x}, t)$ is defined by

$$\boldsymbol{\omega} = \nabla \times \mathbf{u}(\mathbf{x}, t)\,, \tag{2.1}$$

so that immediately $\nabla \cdot \boldsymbol{\omega} = 0$. We can conveniently think of 'vortex tubes' in the flow, i.e. the set of vortex lines passing through any small material surface element δA. The 'circulation' round such a tube is

$$\Gamma = \oint_C \mathbf{u} \cdot d\mathbf{x} = \iint_{\delta A} \boldsymbol{\omega} \cdot \mathbf{n}\, dA\,, \tag{2.2}$$

where C is a closed curve circling the tube once, and this is evidently constant, independent of the particular cross-section of the tube that is chosen (figure 2a). It is frequently stated that vortex lines must either be closed curves or end on a fluid boundary, but this is incorrect: it is now known that in a general three-dimensional flow, the vortex lines are chaotic, and any two neighbouring vortex lines will in general diverge exponentially (a good example may be found in the *'ABC'*–flow studied by Dombre *et al.* (1986)). For this reason, the concept of a vortex tube must be treated with caution, particularly in a turbulent flow in which the cross-section of any instantaneous vortex tube will become seriously deformed if followed far enough along its length.

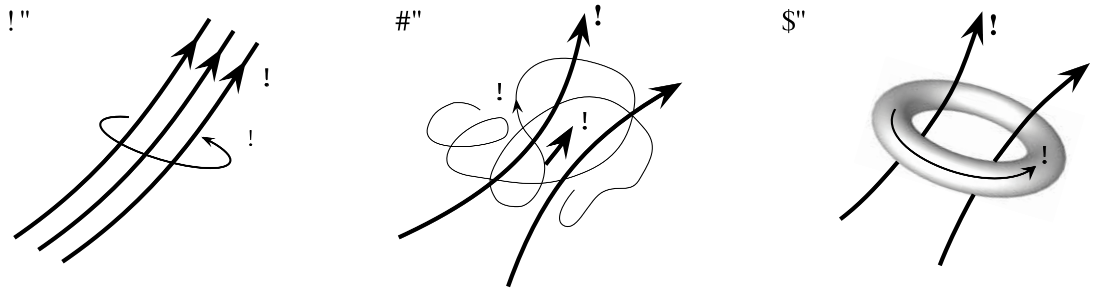

Fig. 2. Vorticity configurations and induced velocity fields. (a) Vortex tube with circulation Γ. (b) Localised vorticity field, and induced velocity, dipolar at a large distance. (c) Vortex ring and its induced velocity.

By virtue of the incompressibility condition $\nabla\cdot\mathbf{u}=0$, we may introduce a vector potential $\mathbf{A}(\mathbf{x},t)$ for $\mathbf{u}$, such that $\mathbf{u}=\nabla\times\mathbf{A},\ \ \nabla\cdot\mathbf{A}=0$. Then we have immediately $\boldsymbol{\omega}=\nabla\times(\nabla\times\mathbf{A})=-\nabla^2\mathbf{A}$. If the vorticity distribution is localised (and by this, we usually mean that $|\boldsymbol{\omega}|$ decreases exponentially rapidly outside some bounded region), then the appropriate solution of this Poisson equation is

$$\mathbf{A}(\mathbf{x},t)=\frac{1}{4\pi}\int\frac{\boldsymbol{\omega}(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}dV'\,. \tag{2.3}$$

The corresponding velocity field is then

$$\mathbf{u}(\mathbf{x},t)=\nabla\times\mathbf{A}=-\frac{1}{4\pi}\int\frac{(\mathbf{x}-\mathbf{x}')\times\boldsymbol{\omega}(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|^3}dV'\,. \tag{2.4}$$

This is the 'Biot-Savart law', giving the velocity field $\mathbf{u}(\mathbf{x},t)$ 'induced' by the vorticity field $\boldsymbol{\omega}(\mathbf{x},t)$. It is this velocity field that transports the vorticity field, a nonlinear feedback that encapsulates the central difficulty of the dynamics of fluids.

If, as supposed, the vorticity field is localised, then for $|\mathbf{x}|>>|\mathbf{x}'|$, (where $\mathbf{x}'$ is any point within the vortical region), equation (2.3) may be manipulated to give

$$\mathbf{A}(\mathbf{x}) \sim -(\boldsymbol{\mu} \times \nabla)\frac{1}{r}\,, \tag{2.5}$$

where

$$\boldsymbol{\mu} = \frac{1}{8\pi}\int \mathbf{x} \times \boldsymbol{\omega}\, dV\,, \tag{2.6}$$

and $r = |\mathbf{x}|$. The corresponding asymptotic behaviour of $\mathbf{u}$ is

$$\mathbf{u} \sim \nabla(\boldsymbol{\mu} \cdot \nabla)\frac{1}{r}\,, \tag{2.7}$$

an irrotational velocity field associated with an (apparent) dipole $\boldsymbol{\mu}$ located at $r = 0$. (The result is independent of the origin chosen for $\mathbf{x}$; proof: an exercise for the reader!) The situation is as sketched in figure 2b. Equation (2.7) shows that the velocity field associated with an arbitrary localised vorticity distribution is dipolar at a large distance, of order r^{-3} as $r \to \infty$.

The most familiar example of a localised vorticity distribution is provided by the 'vortex ring' for which the vorticity field is axisymmetric and confined to a torus, the vortex lines being circles around the axis of the torus (figure 2c). Such vortex rings may be produced and visualised by tapping a smoke-filled box so that air is ejected impulsively through a suitably shaped orifice; both the vortex ring and the smoke are then transported together by the self-induced velocity field. This was the basis of Tait's (1867) demonstration which so impressed Kelvin, who proceeded to calculate the speed of propagation V of a vortex ring of radius R, starting from the Biot-Savart law (2.4), and on the assumption that the vorticity is uniformly distributed across the 'core' of the vortex of small core radius a; his result, recorded in an appendix to Tait (1867), was

$$V = \frac{\omega a^2}{2R}\left(\log\frac{8R}{a} - \frac{1}{4}\right)\,. \tag{2.8}$$

Vortex rings generated by the method of Tait (exploiting the retarding effect of viscosity in the boundary layer inside the orifice) can travel a considerable distance before being dispersed as a result of instability or through the direct action of viscosity. Vortex rings appear to be ubiquitous in nature, the most striking example being the vortex/steam rings emitted in volcanic eruptions (see, for example, the beautiful photographs by Marco Fulle of this phenomenon at `http://www.swisseduc.ch/stromboli/etna/etna00`. A fine example of the persistence of vortex rings (visualised with bubbles at their core), and the playful manner in which dolphins can interact with them can be found at `http://www.metacafe.com/watch/1041454/dolphinplaybubblerings`.

3. The Euler equation and its invariants

We take as a starting point the Navier-Stokes equations for a viscous incompressible fluid in their familiar form

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u}\,, \tag{3.1}$$

$$\nabla \cdot \mathbf{u} = 0\,, \tag{3.2}$$

where ρ is the fluid density (here assumed constant), and ν is the kinematic viscosity of the fluid. If, for the moment, we neglect viscous effects entirely, we simply set $\nu = 0$, giving the equations obtained by Euler (1755).

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho}\nabla p\,, \tag{3.3}$$

$$\nabla \cdot \mathbf{u} = 0\,. \tag{3.4}$$

It is remarkable that, despite the fact that these Euler equations were discovered more that 250 years ago (Eyink *et al.*, 2008), we still do not know whether the solutions that evolve from smooth initial conditions of finite energy remain smooth for all time; or conversely, whether there exist any smooth finite-energy initial conditions for which the solution of the Euler equations becomes singular at finite time. This 'finite-time singularity problem' may seem a rather esoteric issue, of more interest to mathematicians than to geophysicists or engineers; but in fact it lies at the heart of the problem of turbulence, having an obvious bearing on the mechanism of dissipation of energy at the smallest scales of motion, and it is therefore a problem that merits serious study. It is known that, if a singularity occurs at some finite time t_c, say, then the time-integral of the maximum value of the vorticity must diverge as $t \to t_c$ (Beale *et al.*, 1984). This result places the focus of investigation firmly on the behaviour of the vorticity field in general three-dimensional situations. We shall suppose in what follows, that the velocity and vorticity fields do in fact remain smooth for all time, unless otherwise stated.

The Euler equation (3.3) may be written in the equivalent form

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{u} \times \boldsymbol{\omega} - \nabla\left(\frac{p}{\rho} + \frac{1}{2}\mathbf{u}^2\right)\,, \tag{3.5}$$

from which, taking the curl, we immediately obtain the 'vorticity equation'

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \times (\mathbf{u} \times \boldsymbol{\omega}) \,. \tag{3.6}$$

This is the equation that implies that the vortex lines behave like material lines, and are therefore transported with the fluid. Kelvin proved, on the basis of this equation, that the circulation, defined as in (2.2),

$$K = \oint_C \mathbf{u} \cdot d\mathbf{x} \,, \tag{3.7}$$

but now for any material (i.e. 'Lagrangian') circuit C that moves with the fluid, is constant. By virtue of (2.2), K is also the flux of vorticity through C; hence any flow that stretches a vortex tube and (by incompressibility) decreases its cross-section must proportionately intensify the vorticity in the tube. In fact, if $\delta\mathbf{x}$ is an element of a vortex line which moves with the fluid, then $|\boldsymbol{\omega}| \propto |\delta\mathbf{x}|$. [The corresponding result for compressible flow is that $|\boldsymbol{\omega}| \propto \rho|\delta\mathbf{x}|$.]

There are four known invariants of the Euler equations, namely momentum $\mathbf{P}$, angular momentum $\mathbf{M}$, (kinetic) energy E, and helicity $\mathcal{H}$. One might naively suppose that the momentum should be given by $\mathbf{P} = \int \rho\mathbf{u}dV$, the integral being over the whole fluid domain. This integral is however, at best only conditionally convergent, due to the slow $O(r^{-3})$ decrease of $\mathbf{u}$ at infinity. One may calculate the momentum of any given flow by supposing that the corresponding vorticity distribution is established from a state of rest by an impulsive force distribution at the moment under consideration (Saffman, 1995); the result is that

$$\mathbf{P} = \frac{1}{2} \int \rho\mathbf{x} \times \boldsymbol{\omega} \; dV \,, \tag{3.8}$$

an integral that is certainly convergent for any localised vorticity distribution. It may also be verified directly from (3.6) that $\mathbf{P}$ is indeed constant. Note that $\mathbf{P} = 4\pi\boldsymbol{\mu}$, so that the dipole moment of a localised vorticity distribution is constant in time. This result is true also for viscous evolution under the Navier-Stokes equations, the reason being that under the influence of viscosity, momentum is neither created nor destroyed, but merely redistributed by the process of diffusion.

Similarly, the correct expression for angular momentum may be obtained in the form

$$\mathbf{M} = \frac{1}{3} \int \rho\mathbf{x} \times (\mathbf{x} \times \boldsymbol{\omega}) \; dV \,, \tag{3.9}$$

and this integral is also constant under either Euler or Navier-Stokes evolution.

The kinetic energy (divided by density ρ) is given by the convergent integral

$$E = \frac{1}{2}\int \mathbf{u}^2 \, dV\,, \tag{3.10}$$

and this is constant under Euler evolution. However, under Navier-Stokes evolution, we have

$$\frac{dE}{dt} = -\nu \int \boldsymbol{\omega}^2 \, dV\,, \tag{3.11}$$

the right-hand side representing the rate of dissipation of energy by viscosity. The integral on the right is called the 'enstrophy' of the flow, and is usually denoted by the symbol Ω:

$$\Omega = \int \boldsymbol{\omega}^2 \, dV\,, \quad \frac{dE}{dt} = -\nu\Omega\,. \tag{3.12}$$

Like vorticity itself, the enstrophy has a persistent tendency to increase in turbulent flow, a process ultimately controlled by viscosity.

Finally, the helicity $\mathcal{H}$ is given by

$$\mathcal{H} = \int \mathbf{u}\cdot\boldsymbol{\omega} \, dV\,, \tag{3.13}$$

and this also is an invariant of the Euler equations (Moreau, 1961; Moffatt, 1969). Like energy, it is a quadratic functional of the velocity field, but, unlike energy, it is not sign-definite; actually it is a 'pseudo-scalar', changing sign under change from a right- to left-handed frame of reference; this is why we use the non-mirror-symmetric symbol $\mathcal{H}$ to denote it. By the Schwartz inequality, it is bounded in magnitude:

$$|\mathcal{H}| \le E\Omega\,, \tag{3.14}$$

with equality only if $\boldsymbol{\omega}$ is everywhere parallel to $\mathbf{u}$. Such 'Beltrami' flows are evidently flows of maximal helicity. The helicity is conserved even in compressible flows provided these satisfy the barotropic condition that pressure is a function only of density (and not for example of temperature), i.e. $p = p(\rho)$. In fact, helicity is conserved under precisely the same conditions under which Kelvin's circulation theorem is satisfied and vortex lines are frozen in the fluid. The physical interpretation of helicity is topological in character: this integral represents the 'degree of linkage' of the vortex lines of the flow, a quantity that should certainly be preserved under frozen-field evolution . The interpretation is most transparent for the case of two *simply linked* vortex tubes of circulations Γ_1 and Γ_2; for this configuration, it emerges that

$$\mathcal{H} = \pm 2n\Gamma_1\Gamma_2 \,, \tag{3.15}$$

where n is the (Gauss) linking number of the two tubes, and the plus or minus sign is chosen according as the linkage is right- or left-handed (assuming of course, as is conventional, that we use a right-handed frame of reference). This topological interpretation has been extended to flows for which the vortex lines are chaotic (the generic situation) by Arnol'd (1974).

4. The stretched vortex of Burgers (1948)

In a turbulent flow, each constituent vortex tube (or portion of a vortex tube) is subject to the stretching associated with all other vortices in the flow. It is natural therefore to consider an idealised situation in which this stretching is as simple as possible, i.e. axisymmetric, uniform and steady. We consider a vorticity distribution with just one component

$$\boldsymbol{\omega} = (0, 0, \omega(r)) \,, \tag{4.1}$$

where we use cylindrical polar coordinates (r, ϕ, z) with $r^2 = x^2 + y^2$, and we suppose this subjected to the action of 'uniform axisymmetric straining flow' with constant rate of strain $\gamma\,(> 0)$:

$$\mathbf{U} = (-2\gamma r, 0, \gamma z) \,. \tag{4.2}$$

In the absence of this strain, the vortex would diffuse under the action of viscosity; the strain and associated vortex stretching counteracts this effect and a steady state is possible. Note that the additional velocity induced by the vortex is given, from (2.1), by

$$\mathbf{u} = (0, v(r), 0) \,, \tag{4.3}$$

where

$$v(r) = \frac{1}{r}\int_0^r \omega(r')r'\,dr' \,, \tag{4.4}$$

and that this additional velocity has no effect on the vorticity distribution (because $\nabla \times (\mathbf{u} \times \boldsymbol{\omega}) = 0$).

The vortex therefore evolves according to the equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \times (\mathbf{U} \times \boldsymbol{\omega}) + \nu \nabla^2 \boldsymbol{\omega} \,; \tag{4.5}$$

this equation has only a ϕ-component, which reduces to

$$\frac{\partial \omega}{\partial t} = \frac{\gamma}{2r}\frac{\partial (r^2\omega)}{\partial r} + \frac{\nu}{r}\frac{\partial}{\partial r} r \frac{\partial \omega}{\partial r} \,. \tag{4.6}$$

The steady solution, with boundary conditions $\omega(0) = \omega_0$, $\omega \to 0$ as $r \to \infty$, is

$$\omega(r) = \omega_0 \exp -(\gamma r^2/4\nu) \,, \tag{4.7}$$

a gaussian vorticity distribution, with total flux of vorticity

$$\Gamma = 2\pi \int_0^\infty \omega(r) r \, dr = 4\pi\omega_0 \nu/\gamma \,. \tag{4.8}$$

The associated velocity component $v(r)$ is given, from (4.4), by

$$v(r) = \frac{\Gamma}{2\pi r}\left(1 - \exp\left(-\frac{\gamma r^2}{4\nu}\right)\right) \,. \tag{4.9}$$

The circulation round a circle of radius r is $2\pi r v(r)$, and this tends to the constant Γ for $r > \delta$ where $\delta = \nu/\gamma$ is a measure of the radius of the tube. The structure of this vortex is sketched in figure 3.

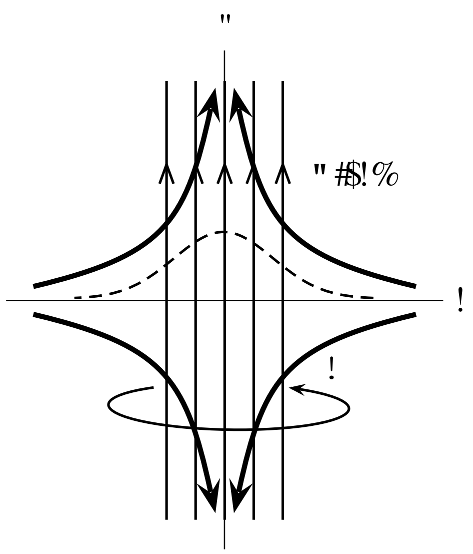

Fig. 3. The stretched Burgers vortex with circulation Γ and gaussian vorticity profile.

A remarkable feature of this vortex, as noted by Burgers (1948), is that the corresponding rate of dissipation of energy per unit length of vortex, namely

$$\Phi = 2\pi\nu \int_0^\infty \omega^2 r \, dr = \Gamma^2\gamma/8\pi \,, \tag{4.10}$$

is independent of ν (for fixed circulation Γ) even in the limit as $\nu \to 0$. In this limit, $\delta \to 0$, $\omega_0 = O(\delta^{-2})$, and the gaussian distribution of vorticity tends to a delta-function. Thus, the vorticity is indeed singular in the limit, yet the rate of dissipation of energy per unit length of vortex remains finite.

If the strain field is non-axisymmetric, of the form

$$\mathbf{U}(x,y,z) = (\alpha x, \beta y, \gamma z), \text{ with } \alpha < \beta \leq 0 < \gamma\,,\ \alpha + \beta + \gamma = 0\,, \quad (4.11)$$

the problem becomes much more complicated, and the behaviour is strongly influenced by the value of the appropriate Reynolds number, here $Re_\Gamma = \Gamma/\nu$. When $Re_\Gamma \gg 1$, as relevant in the context of turbulence, and when $\beta < 0$, the rapid spin within the vortex is sufficient to minimise departures from axisymmetry, and the solution (4.7) is still valid at leading order, the small departures from axisymmetry in the contours of constant ω having an interesting topological structure (Moffatt *et al.*, 1994).

The particular situation when $\beta = 0$ provides a stretched vortex sheet localised near the plane $x = 0$, also with gaussian structure. This two-dimensional solution has been generalised by conformal mapping techniques to provide a wide class of exact solutions of the Navier-Stokes equations exhibiting a fascinating range of 'floral' vortical patterns (Bazant and Moffatt, 2005). For such two-dimensional solutions however, the maximum vorticity in each sheet increases in proportion to $\nu^{-1/2}$ as $\nu \to 0$, and the rate of dissipation of energy per unit area of the vortex sheets is $O(\nu^{1/2})$, thus vanishing in the limit $\nu = 0$, in striking contrast to the axisymmetric case. This is one reason why vortex tubes, rather than vortex sheets, are the more promising candidates for the role of typical structures within a turbulent flow.

5. Kelvin-Helmholtz instability

In consideration of the instabilities to which fluid flows are subject, we should distinguish between 'fast' instabilities, i.e. those that are of purely inertial origin and have growth rates that do not depend on viscosity, and 'slow instabilities', which are essentially of viscous origin, and whose growth rates therefore tend to zero as the viscosity ν tends to zero, or equivalently as the Reynolds number $Re = UL/\nu$ tends to infinity. Examples of fast instabilities are the 'Rayleigh-Taylor instability' that occurs when a heavy layer of fluid lies over a lighter layer, the 'centrifugal instability' (leading to 'Taylor vortices') that occurs in a fluid undergoing differential rotation when the circulation about the axis of rotation decreases with radius, and the

'Kelvin-Helmholtz instability' that occurs in any region of rapid shearing of the fluid. The best known example of a slow instability is the instability of pressure-driven 'Poiseuille flow' between parallel planes, which is associated with subtle effects of viscosity in 'critical layers' near the boundaries; the 'dynamo instability' of magnetic fields in electrically conducting fluids is also diffusive in origin (through magnetic diffusivity rather than viscosity), and may therefore also be classed as a slow instability.

Here, we shall focus on the Kelvin-Helmholtz instability, idealised as the instability of a tangential discontinuity of velocity, which we may take to be

$$\mathbf{U} = (\mp U/2, 0, 0) \quad \text{for} \quad y > \text{or} < 0\,. \tag{5.1}$$

The vorticity is then concentrated on the sheet $y = 0$, and given by the delta-function

$$\boldsymbol{\omega} = (0, 0, U\delta(y))\,. \tag{5.2}$$

We suppose that this sheet is subjected to the sinusoidal perturbation

$$y = \eta(x,t) = \eta(t) \exp ikx\,, \tag{5.3}$$

with $k > 0$, the real part of (5.3) being understood. All perturbations may similarly be supposed proportional to $\exp ikx$. The flow is assumed to be irrotational everywhere except on this disturbed sheet; the perturbation is thus 'isovortical' in the sense that the disturbed vorticity is obtained by a virtual flux-conserving displacement of the undisturbed vorticity field. The velocity above and below the interface then takes the form

$$\mathbf{u} = (-U/2, 0, 0) + \nabla\phi_1 \quad \text{for } y > \eta\,, \tag{5.4}$$

$$\mathbf{u} = (+U/2, 0, 0) + \nabla\phi_2 \quad \text{for } y < \eta\,, \tag{5.5}$$

where, by virtue of incompressibility,

$$\nabla^2\phi_1 = 0 \quad \text{and } \nabla^2\phi_2 = 0. \tag{5.6}$$

Since moreover the perturbation velocity must vanish as $y \to \pm\infty$, it follows that

$$\phi_1 = \Phi_1(t)e^{-ky+ikx}\,, \quad \phi_2 = \Phi_2(t)e^{ky+ikx}\,, \tag{5.7}$$

where $\Phi_1(t)$ and $\Phi_2(t)$ are to be found.

There are now two important conditions that must be satisfied on the vortex sheet $y = \eta(x,t)$. First, since this sheet moves with the fluid, its Lagrangian derivative must vanish, i.e.

$$\frac{D}{Dt}(y - \eta(x,t)) \equiv (\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla)(y - \eta(x,t)) = 0 \quad \text{on } y = \eta\,. \tag{5.8}$$

Now $Dy/Dt \equiv \mathbf{u}\cdot\nabla y = \partial\phi_{1,2}/\partial y$ according as we approach the sheet from above or below. Also, for so long as the disturbance remains small, the problem may be linearised, i.e. squares and products of the small quantities η, Φ_1 and Φ_2 may be neglected and the jump conditions may be applied on $y = 0$ instead of $y = \eta$. It follows that

$$\frac{\partial\phi_1}{\partial y} = \frac{\partial\eta}{\partial t} - \frac{1}{2}U\frac{\partial\eta}{\partial x} \quad \text{and} \quad \frac{\partial\phi_2}{\partial y} = \frac{\partial\eta}{\partial t} + \frac{1}{2}U\frac{\partial\eta}{\partial x} \quad \text{on } y = 0\,. \tag{5.9}$$

Second, the pressure $p = \text{cst} - \rho\partial\phi/\partial t + \rho\mathbf{u}^2/2$ must be continuous across $y = \eta$, so that on linearising,

$$\frac{\partial\phi_2}{\partial t} - \frac{\partial\phi_1}{\partial t} + \frac{1}{2}U\left(\frac{\partial\phi_2}{\partial x} + \frac{\partial\phi_1}{\partial x}\right) = 0 \quad \text{on } y = 0\,. \tag{5.10}$$

Equations (5.9) and (5.10) may now be combined to give, after some simple algebra, the amplitude equation

$$\frac{\partial^2\eta}{\partial t^2} = \frac{1}{4}k^2U^2\eta\,, \tag{5.11}$$

with exponential solutions $\eta \propto e^{\sigma t}$ where $\sigma = \pm kU/2$. Thus the mode for which

$$\sigma = +kU/2 \tag{5.12}$$

grows exponentially until the linearised theory ceases to be valid. These modes (for varying wave-number k) are unstable, and the growth rate is proportional to k, increasing as the wave-length $2\pi/k$ of the disturbance decreases.

The physical mechanism of this instability is that the local strength of the perturbed vortex sheet, given for the unstable mode by

$$\Gamma(x,t) = U + \frac{\partial\phi_2}{\partial x} - \frac{\partial\phi_1}{\partial x} = U + 2i\frac{\partial\eta}{\partial t} = U + ikU\eta\,, \tag{5.13}$$

is $\pi/2$ out of phase with η; the perturbation vorticity is maximal at the points of inflexion where the slope of η is positive, and the induced velocity is such as to amplify the perturbation (figure 4).

This interpretation of the instability mechanism actually continues into the nonlinear regime, investigated by Moore (1979). Moore noted first that, even on linear theory, some kind of singular behaviour is to be expected after a finite time. For, by way of example, suppose that the initial disturbance is periodic in x with period λ, with convergent Fourier series of the form

$$\eta(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{\lambda}\,, \tag{5.14}$$

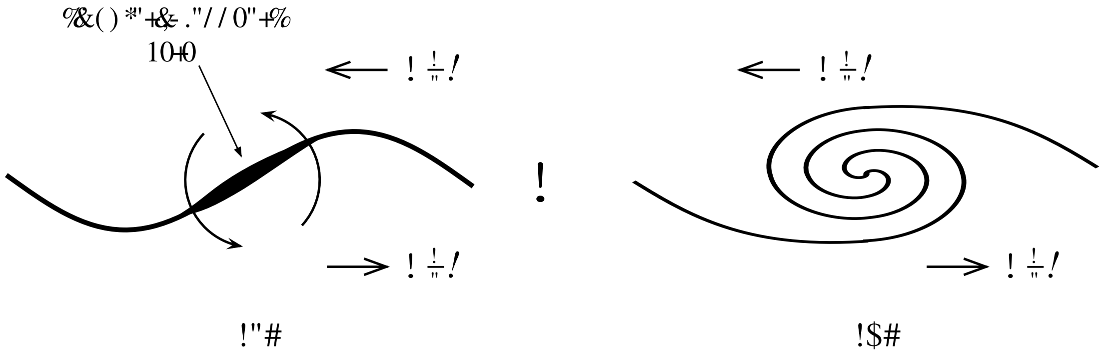

Fig. 4. The Kelvin-Helmholtz instability of a vortex sheet. (a) Vorticity accumulates in the sheet at the upward sloping inflexion points. (b) Spiral wind-up after the Moore singularity.

where

$$A_n = e^{-n} n^{-p}\,, \tag{5.15}$$

with $p > 0$. Thus $\eta(x,t)$ and all its x-derivatives exist at time $t = 0$. However, by virtue of (5.12), selecting only the unstable modes, the disturbance at time t is given by

$$\eta(x,t) = \sum_{n=1}^{\infty} A_n \exp\frac{n\pi U t}{2\lambda}\sin\frac{n\pi x}{\lambda}\,, \tag{5.16}$$

and this series diverges for $t > t_c = 2\lambda/\pi U$, because the exponential growth of the coefficients then defeats the power-law decay for large n.

Now nonlinear effects generate harmonics of the initial disturbance even when this consists of a single Fourier mode, so that a series of the form (5.14) is soon established. Moore's achievement was to show that the exact nonlinear solution for $\eta(x,t)$ becomes singular at a finite time of order λ/U at the upward-sloping inflexion points where, as indicated above, the accumulation of vorticity becomes more and more concentrated. This singularity appears as a discontinuity of curvature, and the vortex sheet strength is cuspidal in form. Beyond the singularity time, observation suggests that the sheet rolls up in a periodic sequence of spiral vortices (figure 4b), although no analytical solution is as yet available to describe this behaviour.

What is important here is that any vortex sheet is absolutely unstable, with a tendency to break up into a series of concentrations of vorticity, more like vortex tubes than a vortex sheet. The vortex tube appears in general to be a much more robust structure than the vortex sheet which has at best a transitory existence, even in turbulent flows.

The Kelvin-Helmholtz instability, as described above, occurs not only for vortex sheets, but also for parallel shear flows having an inflexion point in the velocity profile; the 'tanh' profile

$$\mathbf{U} = (-U/2 \tanh y/\delta, 0, 0)\,, \tag{5.17}$$

for which vorticity is distributed in a layer of thickness $O(\delta)$, is a useful prototype. Such a velocity field is unstable to sinusoidal perturbations of wavelength large compared with δ; on such scales, the velocity profile 'looks like' the discontinuous profile (5.1), so it is not surprising that it exhibits the same type of instability leading to spiral wind-up of the whole vortical layer.

In fact, the existence of at least one inflexion point in the profile of a parallel shear flow of an inviscid fluid is known to be a necessary condition for (linearised) instability of the flow (see, for example, Drazin and Reid (2005)). Plane Poiseuille flow, with its parabolic profile, is therefore stable in the limit of infinite Reynolds number ($\nu = 0$). The source of the (slow) instability of this and similar flows must therefore be sought in the dual role of viscosity, usually thought to be merely stabilising!

6. Transient instability and streamwise vortices

There is however another, potentially more potent, mechanism by which plane parallel non-inflexional flows may be destabilised; this arises through consideration of the shearing of disturbances of finite (rather than infinitesimal) amplitude. Such disturbances, as might be anticipated, can be drawn out into long structures parallel to the flow (or 'streamwise vortices') which, when superposed on the underlying shear flow, provide locally inflexional profiles, which are then subject to the Kelvin-Helmholtz instability. We shall illustrate this behaviour by considering the simplest case of uniform shear flow

$$\mathbf{U} = (\alpha y, 0, 0)\,, \tag{6.1}$$

on which, at time $t = 0$, we superpose a sinusoidal disturbance of the form

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{A}_0 \exp(i\mathbf{k}_0 \cdot \mathbf{x})\,, \tag{6.2}$$

with $\mathbf{k}_0 \cdot \mathbf{A}_0 = 0$ (by incompressibility). For the moment, we retain the effects of viscosity. The analysis that follows was presented by Moffatt (1967), and developed in the context of turbulent shear flow by Townsend (1976).

We suppose that the perturbation, although finite, is still sufficiently weak to allow linearisation of the Navier-Stokes equation:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{U}\cdot\nabla\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{U} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u}\,, \tag{6.3}$$

where p is the perturbation pressure associated with the disturbance. This equation admits a solution of the form

$$\mathbf{u} = \mathbf{A}(t)\exp\left(i\mathbf{k}(t)\cdot\mathbf{x}\right), \quad p/\rho = P(t)\exp\left(i\mathbf{k}(t)\cdot\mathbf{x}\right), \tag{6.4}$$

in which both wave-vector $\mathbf{k}(t)$ and amplitudes $\mathbf{A}(t)$ and $P(t)$ are allowed to vary with time. Such disturbances, first recognised by Lord Kelvin (1887), are known as 'Kelvin modes'. We may note that for a single mode of this kind, the omitted nonlinear term $\mathbf{u}\cdot\nabla\mathbf{u}$ in (6.3) is in fact identically zero, so that (6.4) can provide an exact solution of the Navier-Stokes equation. However, a superposition of modes of different wave-vectors do involve significant nonlinear interactions, which we do not consider here.

Substituting (6.4) in (6.3) gives

$$\dot{\mathbf{A}} + i(\dot{\mathbf{k}}\cdot\mathbf{x})\mathbf{A} + \alpha A_2(1,0,0) + i\alpha y k_1\mathbf{A} = -i\mathbf{k}P - \nu k^2\mathbf{A}\,, \tag{6.5}$$

and we have also, by incompressibility,

$$\mathbf{k}(t)\cdot\mathbf{A}(t) = 0\,. \tag{6.6}$$

The coefficients of x, y and z in (6.5) must vanish; hence $\dot{k}_1 = 0$, $\dot{k}_2 = -\alpha k_1, \dot{k}_3 = 0$, so that

$$k_1 = k_{01}\,, \quad k_2(t) = k_{02} - \alpha k_1 t\,, \quad k_3 = k_{03}\,. \tag{6.7}$$

This simply describes the shearing of the wave fronts, which become more and more aligned parallel to the plane $y = 0$. If $k_1 = 0$, then the wave vector $(0, k_2, k_3)$ remains constant, whereas if $k_1 \neq 0$, then the effect of the shear is asymptotically to align the wave vector in the (0, 1, 0) direction and to increase its magnitude linearly with time.

Here we may note immediately that the effect of the viscous term is simply to introduce a factor

$$\exp\left[-\nu\int_0^t (\mathbf{k}(t))^2 dt\right] = \exp\left[-\nu(k_0^2 t - k_1 k_{02}\alpha t^2 + k_1^2\alpha^2 t^3/3)\right], \tag{6.8}$$

where $k_0 = |\mathbf{k}_0|$, so that, provided $k_1 \neq 0$, this Kelvin mode experiences 'accelerated decay' on a time-scale

$$\alpha t = O(\alpha/\nu k_1^2)^{1/3}\,. \tag{6.9}$$

Modes for which k_1/k_0 is small survive for a long time (when ν is small); the exceptional modes for which $k_1 = 0$ survive for the much longer time-scale $O(1/\nu k_0^2)$, unaffected by the shear. It is the decay of all modes as described by (6.8) that accounts for the stability of the flow $\mathbf{U}$ on linearised analysis. However, before this ultimate decay sets in, the amplitude $|\mathbf{A}(t)|$ may increase by an arbitrarily large factor, as we shall now show.

Noting first, from (6.6), that $\dot{\mathbf{k}} \cdot \mathbf{A} + \dot{\mathbf{A}} \cdot \mathbf{k} = 0$, we have, from (6.5),

$$-ik^2 P = -\dot{\mathbf{k}} \cdot \mathbf{A} + \alpha A_2 k_1 = 2\alpha A_2 k_1 \,, \tag{6.10}$$

and the part of (6.5) not involving x, y and z is then satisfied provided

$$\dot{\mathbf{A}} + \alpha A_2 (1, 0, 0) = -i\mathbf{k}P = 2\alpha A_2 k_1 \mathbf{k}/k^2 \,. \tag{6.11}$$

Integration of the second component of this equation, then of the first and third components, is straightforward; with the notation

$$l^2 = k_1^2 + k_3^2 \,, \quad \tan\theta = l/k_2(t) \,, \quad [\psi] = \psi(t) - \psi(0) \,, \tag{6.12}$$

the solution is

$$A_1(t) = A_{01} - A_{02} \left\{ \frac{k_0^2 k_3^2}{k_1 l^3} [\theta] + \frac{k_1 k_0^2}{l^2} \left[\frac{k_2}{k^2} \right] \right\} , \tag{6.13}$$

$$A_2(t) = A_{02} k_0^2 / k^2 \,, \tag{6.14}$$

$$A_3(t) = A_{03} + A_{02} \frac{k_3 k_0^2}{l^3} \left\{ [\theta] + l \left[\frac{k_2}{k^2} \right] \right\} . \tag{6.15}$$

These three components are plotted in figure 5 for the initial conditions $\mathbf{k}_0 = (0.1, 1, 1)$ and $\mathbf{A}_0 = (1, 1, -1.1)$, for which $k_1/k_0 \approx 0.07$, small enough for there to be a relatively long period of approximately linear growth of $|A_1(t)|$. This period of linear growth increases as k_1/k_0 decreases. The linear growth, or 'transient instability', results from the $(\mathbf{u}\cdot\nabla)\mathbf{U} = u_2 \partial\mathbf{U}/\partial y$ term in equation 6.3, which corresponds to persistent transport of mean-flow x-momentum in the y-direction.

If a random superposition of modes with isotropically distributed initial wave-vectors $\mathbf{k}_0$ is subjected to the above shearing, then the dominant contribution to the disturbance energy will ultimately come from modes with wave-vectors in an increasingly narrow neighbourhood of the plane (in wave-number space) $k_1 = 0$, i.e. from modes for which $\mathbf{k}_0 \cdot \mathbf{U} \approx 0$. Physically this corresponds to the emergence of structures having little or no variation in the streamwise direction. Such structures are known, for obvious reasons, as 'streamwise vortices'; they grow in strength, under the action of the mean shear, until the appearance of inflexion points in the

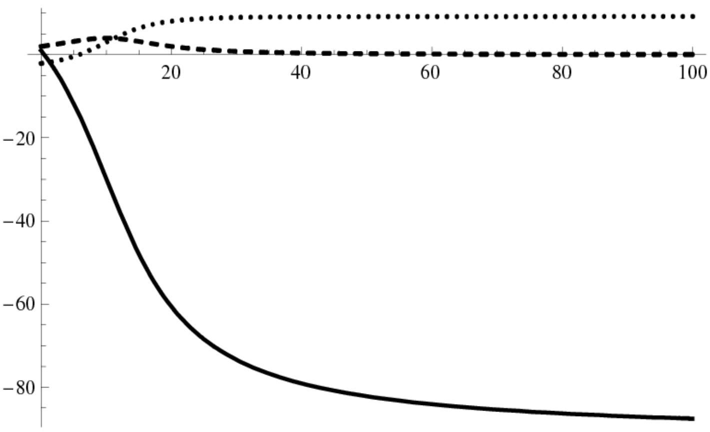

Fig. 5. Evolution of $A_1(t)$ (solid curve), $A_2(t)$ (dashed), and $A_3(t)$ (dotted), as given by (6.13)-(6.15), with initial conditions $\mathbf{k}_0 = (0.1, 1, 1)$ and $\mathbf{A}_0 = (1, 1, -1.1)$ (so $\mathbf{k}_0 \cdot \mathbf{A}_0 = 0$); note the relatively long period of linear growth of $A_1(t)$, a symptom of transient instability.

profile of the total x-component of velocity is inevitable. At that stage the flow is prone to 'secondary instability' of Kelvin-Helmholtz (K-H) origin; the flow becomes fully three-dimensional, and the transition to turbulence is well underway. All this applies of course only if the viscosity parameter ν is sufficiently weak.

The theory described above is a particular case of what is known as 'Rapid Distortion Theory' (RDT), which more generally describes the linearised uniform distortion of a field of turbulence by a mean velocity field of the form

$$U_i(\mathbf{x}) = c_{ij} x_j , \tag{6.16}$$

of which (6.1) is obviously a special case. Such flows may be either elliptic or hyperbolic in character. It is possible to incorporate additional effects relevant in geophysical applications, e.g. uniform density stratification and/or coriolis effects associated with the Earth's rotation. Such effects have been explored in detail by Sagaut and Cambon (2008), where extensive references to previous work on RDT may be found.

It is also worth noting that transient instabilities, as described above, and as greatly developed by Schmid and Henningson (1994), play an important part in more recent work in which new steady and travelling-wave solutions of the classical problems of Couette flow and Poiseuille flow in a pipe have been found. The essential idea (see, for example, Waleffe (2003); Pringle and Kerswell (2007)) is that coherent structures formed by transient instability are unstable to K-H–type instability, and that these (secondary) instabilities interact coherently in such a way as to regenerate the original finite-amplitude perturbations to the flow. The highly original new ideas and results in this area, which have a bearing on the important problem of transition to turbulence, are among the most exciting to emerge in recent years.

7. Turbulence, viewed as a random field of vorticity

Over the last twenty years, turbulence has been increasingly subjected to Direct Numerical Simulation (DNS), i.e. computational treatment of the Navier-Stokes equations without approximation, by either finite-difference or spectral techniques, and 'post-processing' of the numerical output. Figure 6 shows the vorticity distribution in high vorticity regions of a field of turbulence, from a 'state-of-the-art' simulation on the Earth Simulator (Yokokawa *et al.*, 2002); what is important to note here is the apparent 'tube-like' structure of this random field. We referred in the introduction to the persistent stretching of vortex lines in a turbulent flow. Figure 6 gives some substance to this description: each vortex tube is subject to stretching associated with the induced velocity of the whole vorticity distribution (possibly dominated by that of neighbouring vortices), in a manner reminiscent of the Burgers' vortex model of §4 above.

Of course such a description presupposes that there is indeed a systematic stretching effect (rather than the opposite – a systematic contraction). This stretching arises from a natural tendency for any two fluid particles, initially close together, to move apart under the action of a random incompressible velocity field. Indeed, if $\delta\mathbf{x(t)}$ is the separation of two particles, with $\delta\mathbf{x}(0) = \delta\mathbf{a}$ assumed infinitesimally small and non-random, then it can be shown (Orszag, 1977) that in homogeneous, isotropic turbulence (i.e. turbulence whose statistical properties are invariant under translation and rotation)

$$\left\langle \delta\mathbf{x}^2 \right\rangle \geq \delta\mathbf{a}^2 \,. \tag{7.1}$$

When coupled with an assumption concerning the 'finite memory' of turbulence (which amounts to assuming that the turbulence field for times greater than $t + t_c$ is uncorrelated with that at time t), this is sufficient to establish that $\left\langle \delta\mathbf{x}^2 \right\rangle$ increases systematically in time (Davidson, 2004)

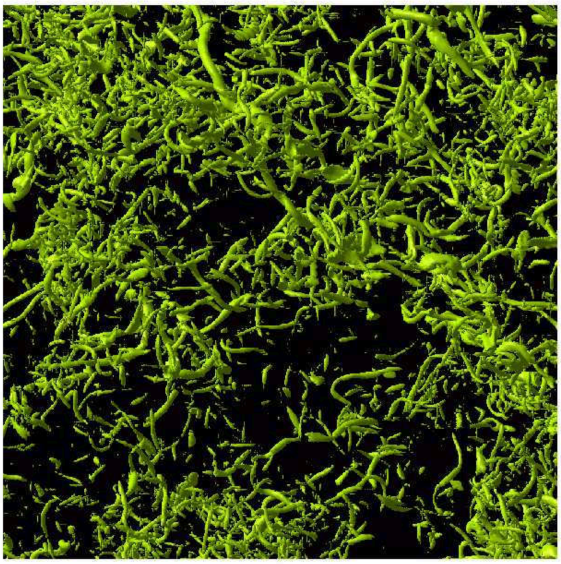

Fig. 6. Intense-vorticity iso-surfaces ($|\omega| > <\omega> +4\sigma$, where σ is the standard deviation of $|\omega|$), in a direct numerical simulation of homogeneous turbulence [from Yokokawa *et al.* (2002), by permission]; this simulation was carried out in a periodic box with 4096^3 grid points, and at a Reynolds number $Re_\lambda = 732$; this Reynolds number is $O(Re^{1/2})$, where $Re = u_0L/\nu$. This figure shows a 'zoomed-in' high vorticity region of size $(748^2 \times 1496)l_v$, where l_v is the 'inner' Kolmogorov scale. Vorticity fluctuations down to this scale are reasonably well resolved.

In particular, if $\delta\mathbf{x}$ is aligned with a vortex line, this element of the vortex line will be systematically stretched by the flow (and this applies to every element of every vortex line!).

The essential ingredients of the dynamics of turbulence may thus be thought of as a combination of three elements: formation of sheet-like structures by shearing of random vorticity (the transient instability mechanism); all-pervasive Kelvin-Helmholtz instability of such structures leading to tube-like structures with possibly some remnants of spiral wind-up; and persistent stretching of such vortices by the strain induced by the surrounding vorticity field. Each of these ingredients has a tendency to decrease the scale of the velocity field, i.e. to contribute to the energy cascade towards the smallest scales of the turbulence, a fundamental aspect of the problem to which we now turn.

8. The Kolmogorov-Obukhov energy-cascade theory

The random character of a turbulent velocity field necessitates a statistical treatment in which an 'ensemble average' $\langle\ldots\rangle$ can be defined. By 'homogeneous' turbulence, as indicated above, we mean turbulence for which all such averages are invariant under translation, i.e. independent of the origin of the coordinate system adopted. By 'isotropic' turbulence, we mean turbulence that is homogeneous and, in addition, invariant under rotation of the frame of reference, i.e. statistically 'the same in every direction'. We note that, if homogeneous turbulence is subjected to uniform strain of the form (6.16), then it remains homogeneous, but develops increasingly marked anisotropy, even if isotropic initially. Homogeneous turbulence has been intensively studied since the pioneering investigations recorded by Batchelor (1953). A modern treatment of the subject, with emphasis on the Kolmogorov (1941) theory and its later modifications, is provided by Frisch (1995).

We restrict attention here to the situation when the mean velocity vanishes: $\langle \mathbf{u} \rangle = 0$. Then attention must be focussed on correlations such as

$$R_{ij}(\mathbf{r}) = \langle u_i(\mathbf{x}) u_j(\mathbf{x}+\mathbf{r}) \rangle \, , \quad S_{ijk}(\mathbf{r}) = \langle u_i(\mathbf{x}) u_j(\mathbf{x}) u_k(\mathbf{x}+\mathbf{r}) \rangle \, , \ldots , \tag{8.1}$$

in standard suffix notation. Equations for such correlation tensors can be obtained from the Navier Stokes equations in a straightforward way; the trouble is that, due to the nonlinearity of these equations, the equation for $\partial R_{ij}/\partial t$ involves terms like $S_{ijk}(\mathbf{r})$; more generally, the time derivative of any nth-order correlation inevitably involves the current value of $(n+1)$th order correlations. This is the famous 'closure problem' that bedevils the subject. No completely satisfactory 'closure' hypothesis (providing an instantaneous relationship between nth-order correlations and those of lower order) has yet been found.

There is however one equation for a second-order quantity that does not involve higher-order quantities[a]: this is the energy equation, easily derived from (3.1):

$$\frac{d}{dt}\frac{1}{2}\left\langle \mathbf{u}^2 \right\rangle = -\nu \left\langle \boldsymbol{\omega}^2 \right\rangle + \epsilon \, . \tag{8.2}$$

The nonlinear term of (3.1) makes no contribution to this energy equation, because it simply redistributes energy over an ever-increasing range of length-scales (as if through the generation of harmonics and subharmonics). We include a term ϵ in (8.2), representing the rate of input

[a]There is also a similar equation for the mean helicity which involves a dissipative term $-\nu < \boldsymbol{\omega} \cdot \nabla \times \boldsymbol{\omega} >$; however, since helicity is not sign-definite, positive helicity generation at one scale can be compensated by negative helicity generation at another, even neglecting the effect of viscosity. This means that the concept of a 'helicity cascade' must be treated with caution.

of energy to the turbulence by some stirring mechanism on a scale L; on dimensional grounds, the level of turbulent energy generated is then of order

$$u_0^2 \equiv \left\langle \mathbf{u}^2 \right\rangle \sim (\epsilon L)^{2/3}\,, \tag{8.3}$$

and we assume that

$$Re = u_0 L/\nu \gg 1\,. \tag{8.4}$$

Under statistically steady conditions, from (8.2),

$$\left\langle \boldsymbol{\omega}^2 \right\rangle = \epsilon/\nu\,, \tag{8.5}$$

from which we note immediately that the enstrophy $\left\langle \boldsymbol{\omega}^2 \right\rangle \to \infty$ as $\nu \to 0$.

The picture then, as conceived by Richardson (1926) and formalised by Kolmogorov (1941), is that energy cascades at a rate ϵ from scales of order L down to scales of order $l_v (\ll L)$ at which viscous effects can dissipate the energy (to heat). The only dimensional parameters on which the scale l_v can depend are ϵ and ν, and it therefore follows on dimensional grounds that

$$l_v \sim (\nu^3/\epsilon)^{1/4}\,. \tag{8.6}$$

It then follows that

$$l_v/L \sim Re^{-3/4}\,, \tag{8.7}$$

so that there is indeed a wide range of scales between the 'energy injection scale' L and the 'dissipation scale' l_v. It is over this range that the energy cascade can proceed.

Kolmogorov (1941) theory is concerned with the statistical properties of turbulence on scales small compared with L, and he assumed that on such scales, these statistical properties are isotropic and depend only on the parameters ϵ and ν, as well as on the separation variable r. Moreover, if $L >> r >> l_v$ (the 'inertial range' of scales), then the statistical properties do not depend on ν. Thus, for example, the 'second-order structure function' $\left\langle (\mathbf{u}(\mathbf{x}+\mathbf{r}) - \mathbf{u}(\mathbf{x}))^2 \right\rangle$ must, on dimensional grounds, have the behaviour

$$\left\langle (\mathbf{u}(\mathbf{x}+\mathbf{r}) - \mathbf{u}(\mathbf{x}))^2 \right\rangle \sim (\epsilon\, r)^{2/3}\,. \tag{8.8}$$

Similarly, the mean-square separation of two fluid particles $\left\langle (\Delta\mathbf{x})^2 \right\rangle$ must increase like

$$\left\langle (\Delta\mathbf{x})^2 \right\rangle \sim \epsilon\, t^3\,, \tag{8.9}$$

for so long as this quantity remains within the inertial range, a result foreshadowed by Richardson (1926) in an early study of atmospheric diffusion.

This is more rapid than conventional diffusion in three dimensions with diffusivity D, namely $\langle(\Delta\mathbf{x})^2\rangle \sim 6Dt$, because, as the particles separate, eddies on progressively larger scales contribute to the diffusive process.

An equivalent formulation of the energy cascade in wave-number space (Obukhov, 1941) gives a result for the energy spectrum function $E(k)$ equivalent to (8.8), namely

$$E(k) = C\epsilon^{2/3}k^{-5/3} \quad (L^{-1} \ll k \ll k_v = l_v^{-1})\,. \tag{8.10}$$

This function $E(k)$ is defined in such a way that

$$\langle(\mathbf{u}(\mathbf{x})^2\rangle = 2\int_0^\infty E(k)\,dk\,, \tag{8.11}$$

so that $E(k)\,dk$ is the contribution to the mean kinetic energy from wave-numbers in the spherical shell $\{k, k+dk\}$ in wave-number space. According to the theory, the dimensionless constant C should be the same in all fields of turbulence, irrespective of the nature of the source of energy on scales of order L, and irrespective of the context, whether environmental, meteorological, astrophysical, or whatever. The first convincing evidence for a $k^{-5/3}$ spectral range came from measurements of turbulence at a Reynolds number of order 10^8 in the tidal channel to the east of Vancouver Island by Grant *et al.* (1962). Since then, the Kolmogorov theory (sketched schematically in figure 7 has provided the bedrock of our understanding of turbulence.

Yet all was not well with the theory, as Kolmogorov (1962) himself recognized; for the rate of dissipation of energy is itself a function of position and time: $\epsilon = \epsilon(\mathbf{x}, t)$, and in regions where $\epsilon > \langle\epsilon\rangle$, the energy cascade presumably proceeds more vigorously, a runaway effect that is now known to generate 'intermittency' in a field of turbulence, i.e. regions of relatively intense vorticity imbedded in more quiescent regions, very much as revealed by DNS. Although intermittency has at most a weak effect on the second-order structure function and on the energy spectrum function (the $k^{-5/3}$-law being apparently quite robust), higher-order statistics are more seriously affected, and the conceptual basis for the Kolmogorov theory is seriously undermined. Huge research effort has been devoted to the problem of intermittency (see, for example, Frisch (1995)), but it seems fair to say that the phenomenon still poses a great challenge to theoreticians.

A further great challenge that remains concerns the behaviour in the 'dissipation range' of wave-numbers $k \sim k_v$ and greater, where $k_v = l_v^{-1} = (\epsilon/\nu^3)^{1/4}$. Here the experimental evidence is that $E(k)$ decays exponentially for $k > k_v$, implying smoothness of the velocity field at the smallest scales (always of course within the limits of a continuum description). On the other

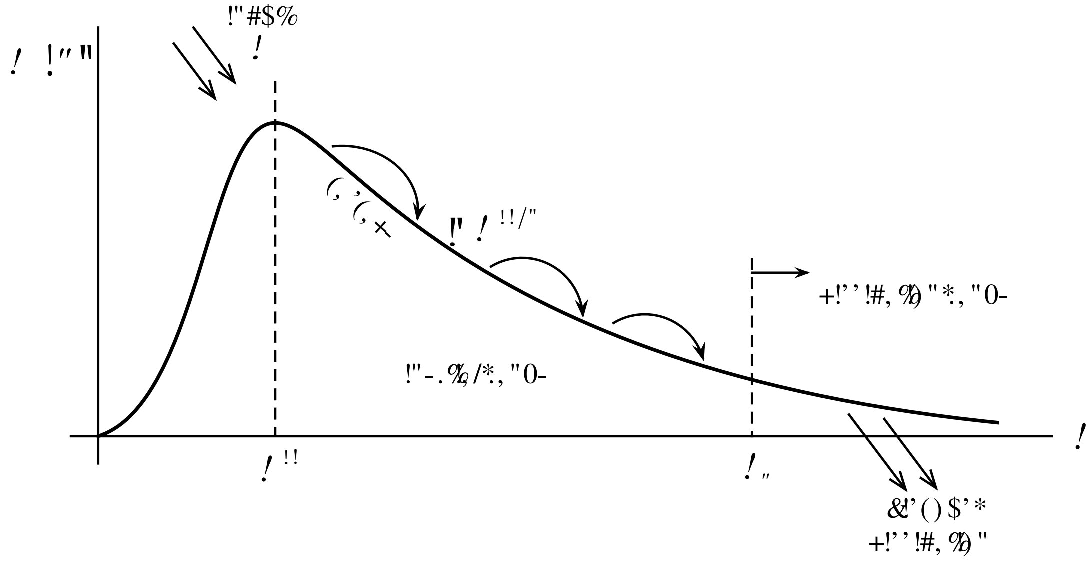

Fig. 7. Energy cascade according to the Kolmogorov-Obukhov scenario; energy is supplied to the turbulence at a rate ϵ on scales of order L, and is dissipated at wave-numbers of order $k_v = (\epsilon/\nu^3)^{1/4}$; for wave-numbers in the inertial range $L^{-1} \ll k \ll k_v$, the energy spectrum function follows a $k^{-5/3}$ power law.

hand, we have the result (8.5) implying the divergence of enstrophy as $\nu \to 0$. This brings us back to the problem posed at the outset of precisely how the energy of turbulence is dissipated at the smallest scales. The Burgers model of section 4 provides an important clue and starting point, but the crucial problem of the *interaction of skewed vortices*, as detected in DNS, remains of central importance at these smallest scales. We may note that, at a Reynolds number of order 10^8 as in the Vancouver tidal channel, if $L \sim 1\,\mathrm{km}$, then $l_v \sim Re^{-3/4}L \sim 1\,\mathrm{mm}$; this range of scales from kilometres down to millimetres in a $3D$ field of turbulence is far beyond what can be simulated in even the most powerful supercomputers of the current era; hence the continuing need for theoretical analysis of turbulence in parallel with experimental observation and carefully crafted numerical simulation.

In this brief introduction to the huge subject of vortex dynamics and turbulence, we have only been able to scrape the surface. Many books are now available for students wishing to pursue the subject in depth. Notable among these is the two-volume encyclopedic work of Monin and Yaglom (1975). The more recent volumes of Davidson (2004) and Sagaut and Cambon (2008) bear testimony to the continuing vitality of the subject. These and other books are distinguished by two asterisks (**) in the list of references that follows.

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