HYPER-LAGRANGIANS AND JOYCE STRUCTURES
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1. Introduction and Background

We will discuss complex hyper-Kähler metrics with extra symmetry, with view to Joyce structures. The main aim of the talk is to describe what they look like in local coordinates.

Joyce structures were introduced by Bridgeland [3] and may be viewed [5] as complex hyper-Kähler metrics with extra symmetries expected to exist on $X = TM$ where M is a space of stability conditions of a triangulated CY_3-category C satisfying some technical assumptions. A Joyce structure has a distinguished adapted coordinate system, the *period coordinates* (z^i, θ^i) for X. The metric is written in terms of a scalar potential satisfying a system of second order non-linear PDEs. An adapted coordinate system of this form for a complex hyper-Kähler metric was first written down by Plebański (in the case of a holomorphic metric on a four-dimensional complex manifold) [10].

We will impose various symmetries on the metric motivated by Joyce structures and see what happens to the equations underlying the hyper-Kähler structure. Some of these symmetries were axiomatised in the definition of a Joyce structure in [5]. The other symmetries we impose are motivated by known constructions of Joyce structures [1],[4],[7].

2. Complex hyper-Kähler metrics

We will be concerned with *complex* hyper-Kähler metrics on a complex manifold X of dimension $4n$. These are holomorphic non-degenerate sections g of $\odot^2 T^*X$ where TX denotes the holomorphic tangent bundle of X, with holomorphic endomorphisms I, J, K of TX satisfying

$$
\begin{aligned}
I^2 = J^2 = K^2 = IJK &= -\operatorname{Id}_{TX}, \\
I^*g = J^*g = K^*g &= g \\
\nabla I = \nabla J = \nabla K &= 0.
\end{aligned}
\tag{2.1}
$$

This differs from (Riemannian) hyper-Kähler geometry and there is no notion of signature.

Define endomorphisms $N := J - iK$ and $N_\infty := J + iK$. The distributions $\ker N$ and $\ker N_\infty$ define complementary foliations of rank $2n$. Let $M := X/\ker N$ and $M_\infty := X/\ker N_\infty$. We call these the *twistor fibres* at zero and infinity respectively. Then $\Omega_0 = g(M, \cdot)$ pushes-down to a symplectic form on M.

As a note of notation let η_{ij} be the standard symplectic matrix given by

$$
\eta = \begin{bmatrix} 0 & \operatorname{Id}_n \\ -\operatorname{Id}_n & 0 \end{bmatrix}
$$

and $\eta_{ik}\eta^{jk} = \delta^i_j$.

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Proposition 2.1 (Plebański normal form). *Given any Darboux coordinates z^i for M so that*

$$\Omega_0 := \eta_{ij} dz^i \wedge dz^j \tag{2.2}$$

then there exist coordinates θ^i so that (z^i, θ^i) give coordinates for X and

$$g = \eta_{ij} d\theta^i \odot dz^j + \frac{\partial^2 P}{\partial\theta^i \partial\theta^j} dz^i \odot dz^j. \tag{2.3}$$

with

$$\frac{\partial^2 P}{\partial\theta^i \partial z^j} - \frac{\partial^2 P}{\partial\theta^j \partial z^i} - \eta^{kl} \frac{\partial^2 P}{\partial\theta^i \partial\theta^k} \frac{\partial^2 P}{\partial\theta^j \partial\theta^l} = 0. \tag{2.4}$$

and furthermore setting

$$U_j = \frac{\partial}{\partial z^j} + \eta^{kl} \frac{\partial^2 P}{\partial\theta^j \theta^k} \frac{\partial}{\partial\theta^l} \tag{2.5}$$

$$V_j = \frac{\partial}{\partial\theta^j} \tag{2.6}$$

we have $J(U_j) = V_j$ and $K(U_j) = iV_j$.

(2.4) is natural generalisation of Plebański's second heavenly equation to $4n$ dimensions. When $n = 1$ it is the reduction of the vacuum Einstein equation for an anti-self-dual holomorphic metric. We call such coordinates (z^i, θ^i) *Plebański coordinates* for the complex hyper-Kähler structure.

3. Joyce structures

For our purposes a Joyce structure will be a manifold M of complex dimensions $2n$ equipped with:

- a holomorphic symplectic form ω,
- Darboux coordinates z^i,
- a complex hyper-Kähler structure on $X = TM$ such that $\pi^*\omega = \Omega_0$, where π is the canonical projection

satisfying the following symmetries

1. The canonical tangent coordinates (z^i, θ^i) are Plebański coordinates for the complex hyper-Kähler structure.
2. $\mathcal{L}_E g = g$ where $E = z^i \frac{\partial}{\partial z^i}$.
3. g is odd under simultaneously multiplying all the fibre coordinates by -1.
4. g is invariant under translations[1] in the fibres

$$\theta^j \mapsto \theta^j + 2\pi i$$

We call M the *base* of the Joyce structure. M is identified as the space of stability conditions in Joyce structures arising in DT theory.

We can write these conditions in terms of the potential P. Property (3) has the remarkable consequence that

$$g^J := z^i \frac{\partial P^3}{\partial\theta^i \partial\theta^j \partial\theta^k}\bigg|_{\theta=0} dz^j \odot dz^k \tag{3.1}$$

defines a flat metric on M (if non-degenerate), and furthermore, the Levi-Civita connection ∇ of the metric g preserves tangents to the zero section $y^i = 0$ and preserves g^J.

[1] In the literature: e.g. [3],[5],[7] one needs to allow for cases when g is invariant under translations by a more general lattice

In practice many of the examples of Joyce structures we construct have poles and it is a work-in-progress to determine if g^J actually exists and is non-degenerate in these cases. In fact we will see in some cases it vanishes.

Property (4) says that really the Joyce structure lives on a $(\mathbb{C}^*)^{2n}$ bundle over M.

4. Einstein-Weyl?

Symmetry reductions of ASD metrics in four dimensions by (non-null) conformal Killing vectors leads to Einstein-Weyl geometry on the space of orbits [9]. In the hyper-Kähler case these reductions were studied by Dunajski and Tod in [8] and the Euler vector field here, assuming it is non-null, fits into the setting discussed there.

As far as I understand there has been no chracterisation of the geometry on the space of trajectories of a conformal Killing vector of a half-flat metric in higher dimensions. The Einstein-Weyl one-form is defined by taking the Hodge-star of $(\nabla_{[a}E_b)E_{c]}$ so the construction would need to be completely different.

5. Holomorphic quadratic differentials and M_∞

For the large class of Joyce structures in [1] coming from moduli of *holomorphic* quadratic differentials on Riemann surfaces, the Euler vector field E has the property of acting trivially on M_∞. In other words, E should be tangent to $\ker N_\infty$. Writing this out in local coordinates

$$s^j \frac{\partial}{\partial z^j} + s^j \eta^{kl} \frac{\partial^2 P}{\partial \theta^j \partial \theta^k} \frac{\partial}{\partial \theta^l} = z^i \frac{\partial}{\partial z^i}. \tag{5.1}$$

We come to the conclusion that E tangent to $\ker N_\infty$ implies

$$z^i \frac{\partial^2 P}{\partial \theta^i \partial \theta^j} = 0. \tag{5.2}$$

The first thing to note is that the g^J vanishes and E is a null vector: $g(E, E) = 0$. Contracting (2.4) with z^i then yields the linear system (using the homothety condition)

$$\frac{\partial P}{\partial \theta^i} + z^i \frac{\partial^2 P}{\partial \theta^i z^j} = 0. \tag{5.3}$$

If $\dim(X) = 4$ (all of the Joyce structures that come from holomorphic quadratic differentials have $\dim(X) = 8$ or above) and ignoring properties (3) and (4) for the moment, we can just solve for the metric. We obtain

$$g = \eta_{ij} d\theta^i \odot dz^j + \eta_{kp}\eta_{lq} \frac{T\left(\frac{z^1}{z\cdot\theta}, \frac{z^2}{z\cdot\theta}\right)}{(z\cdot\theta)^3} z^k z^l \, dz^p \odot dz^q \tag{5.4}$$

where T is an arbitrary function. The metric is a generalisation of the Sparling-Tod H-space [11]. Enforcing property (4), one may convince themselves via some complex analysis that the metric must be constant with respect to the θ^i. In particular, it is flat.

6. Hyper-Lagrangians

The following objects are a feature of Joyce structures that have been constructed from moduli of quadratic differentials:

Definition 6.1 (Projectable hyper-Lagrangian foliation)**.** *Let (X, g) be complex hyper-Kähler manifold. Then $B \subseteq TX$ is a hyper-Lagrangian foliation if B is a Frobenius integrable distribution with each leaf Lagrangian for $\Omega_I, \Omega_J, \Omega_K$. We say such a foliation is projectable if B pushes-down to a (necessarily Lagrangian) foliation L of M.*

Take $M = \mathrm{Quad}(\mathbb{CP}^1, \{2n+5\})$ as in [7], then each quadratic differential $\phi = Q_0(x)dx^2$ defines a hyperelliptic curve Σ defined by $y^2 = Q_0(x)$ equipped with a meromorphic 1-form $\psi = ydx$. Differentiating ψ gives a map $\mu_0 : T_\phi M \to H^1(\Sigma, \mathbb{C})$. Then B is given by the unique hyper-Lagrangian lifts of $L_\phi = m^{-1}(H^{1,0}(\Sigma, \mathbb{C}))$, $\phi \in M$.

Given such a foliation we can by Proposition **??** take Plebański coordinates (x^i, y^i) adapted to (the generalisation of) the second heavenly equation such that the Lagrangian L is given by holding $x_{n+1}, ..., x_{2n}$ constant (we have freedom to choose such Darboux coordinates on M). Then some algebra shows the unique hyper-Lagrangian distribution in X pushing-down to L is

$$B := \mathrm{span}\left\{\frac{\partial}{\partial y_i}, \frac{\partial}{\partial x_i} + \eta_{jk}\frac{\partial^2 \hat{P}}{\partial y^i \partial y^j}\frac{\partial}{\partial y^k}\right\}_{i=1}^{n} \tag{6.1}$$

where $\hat{P}$ is the Plebański potential (not necessarily the same one!). Integrability implies that

$$\frac{\partial \hat{P}}{\partial y^i \partial y^j \partial y^k} = 0, \quad i, j, k \leq n. \tag{6.2}$$

These adapted coordinates are related to the period coordinates by

$$y^i = \frac{\partial x^i}{\partial z^j}\theta^j. \tag{6.3}$$

Meanwhile the Plebański potentials can be shown to be related by

$$\hat{P} = P + \Phi_{ijk}(x) y^i y^j y^k \tag{6.4}$$

for some functions $\Phi_{ijk}(x)$ on the base. The second derivatives of P with respect to y^i are periodic under lattice transformations and $\hat{P}$ is quadratic in half the fibre coordinates. We will use this to constrain the functional form of $\hat{P}$ considerably.

7. Elliptic functions

In four dimensions we may write

$$\hat{P} = A(x^1, x^2, y^2)y^1y^1 + 2B(x^1, x^2, y^2)y^1 + C(x^1, x^2, y^2).$$

The heavenly equation decouples somewhat

$$2AA_{y^2y^2} - 4A_{y^2}^2 + A_{y^2x^1} = 0 \tag{7.1}$$

$$2AB_{y^2y^2} - 4B_{y^2}A_{y^2} + B_{y^2x^1} - A_{x^2} = 0 \tag{7.2}$$

$$2AC_{y^2y^2} - 4B_{y^2}^2 + C_{y^2x^1} - 2B_{x^2} = 0. \tag{7.3}$$

After a bit of thought, the periodicity of P can be shown to imply that A_{y^2} is an elliptic function of y_1. It is also even by the symmetries of the Joyce structure. It follows it is a rational expression in the Weierstrass $\wp$ function with periods

$$\omega_1 := 2\pi i \frac{\partial x^2}{\partial z^1} \quad \omega_2 := 2\pi i \frac{\partial x^2}{\partial z^2} \tag{7.4}$$

which are functions on M. In the case of the A_2 Joyce structure when

$$M = \mathrm{Quad}(\mathbb{CP}^1, 7)$$

we have written the metric of [4] in these coordinates which will correspond to a solution of the above system. It has

$$A_{y_2} = -\frac{1}{\wp'(y^2; -4x^2, -4x^1)^2}. \tag{7.5}$$

References

[1] T. Bridgeland, *Joyce structures on spaces of quadratic differentials*, arXiv:2203.17148v2 (2024).
[2] T. Bridgeland, *Tau functions from Joyce structures*, arXiv:2303.07061v1 (2023).
[3] T. Bridgeland, *Geometry from Donaldson-Thomas invariants*, Integrability, Quantization, and Geometry II. Quantum Theories and Algebraic Geometry, 1–66, Proc. Sympos. Pure Math. Amer. Math. Soc. (2021).
[4] T. Bridgeland, D. Masoero, *On the monodromy of the deformed cubic oscillator*, Math. Ann. **385**, 193–258 (2023).
[5] T. Bridgeland, I.A.B. Strachan, *Complex hyperkähler structures defined by Donaldson–Thomas invariants*, Lett. Math. Phys. **111**, 54 (2021).
[6] M. Dunajski, *Null Kähler Geometry and Isomonodromic Deformations*, Commun. Math. Phys. **391**, 77–105 (2022).
[7] M. Dunajski, T.Moy, *Heavenly metrics, hyper-Lagrangians and Joyce structures,* arXiv:2402.14352 (2024).
[8] M. Dunajski, K.P. Tod, *Einstein–Weyl structures from hyper–Kähler metrics with conformal Killing vectors*, Differ. Geom. App. **14** 39–55 (2001).
[9] P. Jones, K.P Tod, *Minitwistor spaces and Einstein-Weyl spaces*, Class. Quant. Grav. **2**,4, 565-577 (1985).
[10] J. Plebański, *Some solutions of complex Einstein equations* J. Math. Phys.**16** 2395-2402. (1975).
[11] G. A. J. Sparling, K. P. Tod, *An example of an H-space* J. Math. Phys. **22** 331–332 (1981).